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1996 J. Phys. A: Math. Gen. 29 4561

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J. Phys. A: Math. Gen. **29** (1996) 4561–4587. Printed in the UK

Splitting instability: the unstable double wells*

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Received 12 December 1995, in final form 15 April 1996

Abstract. In this paper we perform the semiclassical analysis of a pair of resonances in the case of a quasi-symmetrical unstable double well. We consider two kinds of asymmetric perturbations: one supported in the infinite external well, the other one of the Stark kind. We prove that the first perturbation is able to localize each state inside one of the internal wells so that we have linear Stark effect and vanishing of the splitting at the crossing point of the two resonances. This phenomenon is critical in the ratio between the internal and external barrier lengths, and the critical value of the ratio is close to two. Possible applications to the molecular structure and to the vanishing of the inversion frequency are briefly discussed.

1. Introduction

Among the beautiful results in semiclassical quantum mechanics we have the double well localization for small perturbations restricted to the barriers [3, 9, 12]. Such localization of a pair of bound states is associated to a small growing of the splitting.

Let us consider the Stark effect for this model with very weak fields (of the order of the splitting). If we plot the graphic of the first two levels as functions of the field strength, we can see that the localization region (for all the parameters) is the one where we have a locally linear (not quadratic) behaviour with respect to the field. But, if we take a field strength sufficiently large to give the crossing of the levels, we lose the localization and actually we have an avoided crossing and a quadratic Stark effect at the crossing point.

Now, considering the molecular structure effect, as for instance in the ammonia case, we observe the vanishing of the splitting (i.e. the inversion frequency) together with the occurrence of localization. Indeed, the experimental data show that the inversion frequency decreases regularly to zero as the environment action (i.e. the gas pressure) increases. Moreover the Stark effect is expected to be linear in this case [15]. Of course, all the explanations of the molecular structure are related to the existence of perturbations due to the environment. Obviously the localized states are not considered to be stationary eigenstates of the symmetrical molecular operator.

The racemization effect shows the instability of the molecules, at least at high temperature [6].

* This work is partially supported by the Italian CNR-GNFM and MURST.

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0305-4470/96/154561+27$19.50 © 1996 IOP Publishing Ltd

A complete model of the molecular structure should take into account both the formation of chains of molecules and the occurrence of molecular collisions.

One particular model concerns the non-linear Stark effect due to the reaction field generated by the asymmetric molecules [1, 2, 4]. In such a model we have localization, but the splitting vanishes in a discontinuous way because of the selection rules. Another simple model was proposed a long time ago by Margenau [11]. It consists in a system of two double wells representing the shape potential of two close molecules. In this case we have two different inversion frequencies, and actually the smaller one is more relevant in the radiation spectrum. This model gives reasonable results for intermediate pressures, but it is not able to give the vanishing of the inversion frequency at high pressure.

We propose here, as a simplified molecular model, a time independent one consisting in an unstable symmetrical double well, or *double volcano*, with small perturbations. The instability comes from the shape of the potential, with an infinite well connected by tunneling to the double well. The shape of the potential suggests the picture of a *double well in an island* [8] or a *double volcano on the ocean*. This model appears as a prototype and a research laboratory of splitting instability in the quasi-symmetrical case. Actually, with such a model, extended to the many dimensional case and using modern techniques of analysis, we obtain all the effects we have in molecules: localization stable with respect to the Stark effect, linear Stark effect and vanishing of the splitting at the crossing point. In order to show what happens in a simpler way, we start with a toy model which can be treated with elementary methods. In this model it is possible to have vanishing of the splitting at a very small electric field (with respect to the splitting), and this happens when the depth of the ocean is of the order of the semiclassical parameter.

In all our models we have a parameter usually considered fixed: the instability one related to the ratio between the internal barrier length and the external one. We show that the results of stable localization, linear Stark crossing and vanishing of the splitting are all dependent on the instability parameter. We fix a critical value of the instability parameter, depending on the beating effect of the double well, above which (that is in the hypercritical case) we obtain stable localization by means of a very small external perturbation. For this value of the instability parameter the inter-well barrier length is twice the external one, and equivalently the mean life of the system is equal to the beating period of the symmetrical problem.

Since the molecular structure represents the persistence of classical notions in microscopical systems (actually non-isolated!), then it is an example of strong instability in the semiclassical limit. In particular we observe instability of delocalization and splitting. The practical result is the small relevance of quantum stationary states and the long persistence of metastable localized states [15]. Since we expect the localized states to be metastable [15], we introduce an instability in the system as given by the environment. In such a way we are able to have both localization and vanishing of the splitting for small asymmetric perturbations if the instability is hypercritical. This could explain the transition from a quantum to a classical behaviour at a finite semiclassical parameter as it happens in nature. Actually, in order to compare with physics, we should prove the existence of this critical effect in time-dependent unstable models.

Now, let us describe better the models and the results.

As a toy model in one dimension we consider a potential with two steps and two delta functions. We take the ocean depth small and a flat perturbation extending throughout one half of the ocean. In this case we have no need of the Stark effect for the vanishing of the splitting at the leading order.

Then, the model of molecular potential we consider consists of a symmetrical unstable double well (i.e. double volcano) potential in any dimension, with enough regularity in order to use the Helffer–Sjöstrand method, or also, when it is possible, the external complex scaling. In this model the molecular metastable states are delocalized in the double well region (i.e. the island). Such a delocalization can disappear when we introduce an exponentially small (with respect to the semiclassical parameter $\hbar$) perturbation which causes the breakdown of the symmetry. For completeness we consider two different kinds of perturbations. The first one, called internal perturbation (section 5.2), is given by means of a potential which is a non-negative C_0^∞ function with its support inside the island, but far from each well. With such a perturbation we are able to extend the localization results of Simon [12] to the unstable case, but we do not obtain the splitting instability results we announced above. The second one, called external perturbation (section 5.3), is able to give all the desired results. It consists in a potential defined by a non-smooth function with compact support contained in the ocean. More precisely, we assume that the external perturbation is given by $W = w\mathbb{1}_{\mathcal{W}}$ where w is a positive C^∞ function and $\mathbb{1}_{\mathcal{W}}$ is the characteristic function of a compact set $\mathcal{W}$ with smooth boundary.

Let us quote our previous paper [5] where we discuss a model having a large asymmetry, but otherwise being similar to the present one. In that paper we discuss the transition from anti-crossing to crossing of Stark effect resonances for growing instability.

The paper is organized as follows: in section 2 we discuss the toy model; in section 3 we give the general notations and we discuss the class of models we consider; in section 4 we briefly recall the definition of resonances given by Helffer and Sjöstrand [8] and we consider the single volcano resonances; in section 5 we consider the double volcano resonances in the absence of asymmetric perturbation (section 5.1), with internal asymmetric perturbation (section 5.2), with external asymmetric perturbation (section 5.3) and with both external perturbation and the Stark effect (section 5.4); in the appendix we consider the perturbation norm.

2. The toy model

As a continuation of the introduction let us discuss a simple model which can be solved explicitly by elementary methods. Since the discussion is very similar to the one of the real model, with similar results, it is useful for a better understanding of the problem. Actually, we now consider a case where the imaginary part of the resonances is more sensitive to the perturbation than its real part. We treat exactly the case in which we have the vanishing of the splitting at very small field strength. In particular we make the depth of the ocean vanish in the semiclassical limit.

Let us consider the Hamiltonian:

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_d(x). \tag{2.1}$$

The symmetrical potential with the Stark perturbation is given by:

$$V_0(x) = -\gamma + (\alpha + \gamma)\mathbb{1}_{[-2-\epsilon,2+\epsilon]}(x) - b_-\delta(x + 1 + \epsilon) - b_+\delta(x - 1 - \epsilon) \tag{2.2}$$

where $b_\pm = \beta(1 \mp \phi)$, $\alpha = \beta^2/4$, $\epsilon, \beta, \gamma \in \mathbb{R}^+$, $\phi \in \mathbb{R}$ and $\delta(x)$ is the Dirac's δ. The perturbed potential is:

$$V_d(x) = V_0(x) + d\mathbb{1}_{[2+\epsilon,\infty)}(x) \tag{2.3}$$

where $\gamma > d$. If we consider the Sommerfeld solutions at $\pm\infty$, respectively, and we match their logarithmic derivatives at the origin, we get an equation for the two resonances:

$$\frac{\kappa g_-(2+\epsilon) - b_- f_-(1)c(1+\epsilon)}{\kappa f_-(2+\epsilon) - b_- f_-(1)s(1+\epsilon)} = -\frac{\kappa g_+(2+\epsilon) - b_+ f_+(1)c(1+\epsilon)}{\kappa f_+(2+\epsilon) - b_+ f_+(1)s(1+\epsilon)} \tag{2.4}$$

where

$$f_\pm(x) = c(x) - \mathrm{i}(k_\pm/\kappa)s(x) \qquad g_\pm(x) = s(x) - \mathrm{i}(k_\pm/\kappa)c(x) \tag{2.5}$$

$$c(x) = \cosh(\kappa x) \qquad s(x) = \sinh(\kappa x) \tag{2.6}$$

and

$$\kappa = \sqrt{\alpha - E} \qquad k_- = \sqrt{E+\gamma} \qquad k_+ = \sqrt{E+\gamma-d}. \tag{2.7}$$

We consider the semiclassical regime for $1/\beta$ small, and we take ϕ exponentially small with respect to the semiclassical parameter $1/\beta$. Thus, we get the approximate equation:

$$\frac{\eta - \gamma_- - \phi - \mu}{\eta - \gamma_- - \phi + \mu} - \frac{\eta - \gamma_+ + \phi - \mu}{\eta - \gamma_+ + \phi + \mu} \sim 0 \qquad \text{as } \frac{1}{\beta} \to 0 \tag{2.8}$$

where:

$$\gamma_\pm = \frac{1 + \mathrm{i}k_\pm/\kappa}{1 - \mathrm{i}k_\pm/\kappa} \sim 1 + \mathrm{i}\frac{4k_\pm}{\beta} \qquad \eta = \left(\frac{2\kappa}{\beta} - 1\right)\mathrm{e}^{2\kappa} \qquad \mu = \mathrm{e}^{-2\epsilon\kappa}. \tag{2.9}$$

The solutions η of the approximate equation are:

$$\eta_\pm = (\gamma_+ + \gamma_-)/2 \pm \sqrt{[(\gamma_+ - \gamma_-)/2 - \phi]^2 + \mu^2}. \tag{2.10}$$

So, we see that $\Re\eta_\pm$ (and the corresponding positions of the resonances $E_\pm \sim -2\alpha\mathrm{e}^{-\beta}\Re\eta_\pm$) as functions of ϕ cross each other at $\phi = 0$ with nearly linear behaviour if $\gamma_+ - \gamma_-$ is nearly imaginary as shown above, and $[\Im(\gamma_+ - \gamma_-)]^2 \gg 4\mu^2$, i.e. if

$$(k_+ - k_-)^2/\beta^2 \sim \left(\sqrt{\gamma} - \sqrt{\gamma - d}\right)^2/\beta^2 \gg 4\exp(-2\beta\epsilon). \tag{2.11}$$

The last inequality is satisfied if ϵ vanishes with the semiclassical parameter, with the condition:

$$\epsilon \gg \ln(\beta)/\beta. \tag{2.12}$$

Thus, even in this simple model we have the critical instability when the external barrier is nearly one half of the internal one (i.e. ϵ small).

3. General notations

We consider the semiclassical Schrödinger operator formally defined on $L^2(\mathbb{R}^n)$:

$$P := -\hbar^2\Delta + V(x) \tag{3.1}$$

where $V(x)$ is a symmetrical unstable double-well potential (simply *double volcano* or *double well* in the following). More precisely, $V(x)$ is a $C^\infty(\mathbb{R}^n, \mathbb{R})$ function such that

$$\limsup_{|x|\to\infty} V(x) < 0 \qquad \text{and} \qquad \{x \in \mathbb{R}^n : V(x) \leqslant 0\} = \{x_1, x_2\} \cup U \tag{3.2}$$

and there exists a symmetry $\mathcal{S}$ with respect to a hyperplane such that

$$[V, \mathcal{S}] = 0. \tag{3.3}$$

The unbounded set U is called *ocean* (or also *infinite external well*) and $\{x_1\}$ and $\{x_2\}$ are called *wells*, the open set Ö:$= \mathbb{R}^n \backslash U$ is called *island*. By means of a suitable choice of coordinates we simply assume that

$$(\mathcal{S}f)(x^1, x^2, \ldots, x^n) = f(-x^1, x^2, \ldots, x^n) \tag{3.4}$$

hence $x_2 = \mathcal{S}x_1$ provided that $\mathcal{S}x_1 \neq x_1$. In the following we assume that the two minima are non-degenerate, that is

$$V(x_\ell) = 0 \qquad \nabla V(x_\ell) = 0 \qquad [\mathrm{Hess}V](x_\ell) > 0 \qquad \ell = 1, 2. \tag{3.5}$$

Now, let

$$P_\nu := P + \nu W \tag{3.6}$$

be the perturbed double well operator where ν is a real parameter and where W is a bounded perturbation which causes the breakdown of the symmetry, i.e.

$$\mathcal{S}W \neq W\mathcal{S}. \tag{3.7}$$

We consider two different classes of perturbations W: *internal* perturbations, where W is a $C_0^\infty(\mathbb{R}^n, \mathbb{R})$ non-negative function with compact support $\mathcal{W}$ disjoint from the wells and such that $\mathcal{W} \cap \ddot{\mathrm{O}} \neq \emptyset$, and *external* perturbations, where $W = w\mathbb{1}_{\mathcal{W}}$, w is a positive $C^\infty(\mathbb{R}^n, \mathbb{R})$ function and $\mathbb{1}_{\mathcal{W}}$ is the characteristic function on a compact set $\mathcal{W}$ contained in U and with smooth boundary, i.e. $\mathbb{1}_{\mathcal{W}}(x) = 1$ if $x \in \mathcal{W}$ and $\mathbb{1}_{\mathcal{W}}(x) = 0$ if $x \notin \mathcal{W}$.

Let

$$S_0 := \rho(x_1, x_2) \tag{3.8}$$

be the Agmon distance between the two wells $\{x_1\}$ and $\{x_2\}$ and let

$$S_\ell := \rho(x_\ell, U) := \inf_{x \in \partial U} \rho(x_\ell, x) \qquad S := S_1 = S_2 \tag{3.9}$$

be the Agmon distance between each well and the ocean U. Let s_ℓ be the Agmon distance between the well $\{x_\ell\}$ and the support of the perturbation:

$$s_\ell := \rho(x_\ell, \mathcal{W}) \qquad \mathcal{W} := \mathrm{supp}\, W \tag{3.10}$$

and let

$$s := \min\{s_1, s_2\}. \tag{3.11}$$

The Agmon (pseudo-)distance on $\mathbb{R}^n$ is defined as

$$\rho(x, y) := \rho_V(x, y) = \inf_\gamma L_\gamma \qquad x, y \in \mathbb{R}^n \tag{3.12}$$

where the infimum is taken on the possible piece-wise C^1 paths γ connecting the two points x and y and where L_γ is the length of the path γ with respect to the Agmon metric $\mathrm{d}s^2 := \max[0, V(x)]_+ \mathrm{d}x^2$.

We consider the following two cases:

S *sub-critical case*: $S_0 < 2S$; thus the minimal geodesics connecting the two wells are contained in the open set Ö;

H *hypercritical case*: there are not minimal geodesics of length less than or equal to $2S$ connecting the two wells and contained in Ö (see figure 1), thus $S_0 = 2S$ for $n > 1$ and $S_0 > 2S$ for $n = 1$.

We don't dwell on the critical case, corresponding to $S_0 = 2S$ and where there exists at least a minimal geodesic connecting the two wells and contained in Ö, even if it can be treated similarly to the hypercritical case.

In the hypercritical case **H** we assume:

$$\overline{B(x_1, S)} \cap \overline{B(x_2, S)} \cap \partial\ddot{\mathrm{O}} = \emptyset \tag{3.13}$$

where

$$B(A, \delta) := \{x \in \mathbb{R}^n : \rho(x, A) < \delta\} \tag{3.14}$$

that is the endpoint of each minimal geodesic, connecting the well $\{x_1\}$ with the ocean U, does not coincide with the one of a minimal geodesic connecting U with the other well $\{x_2\}$.

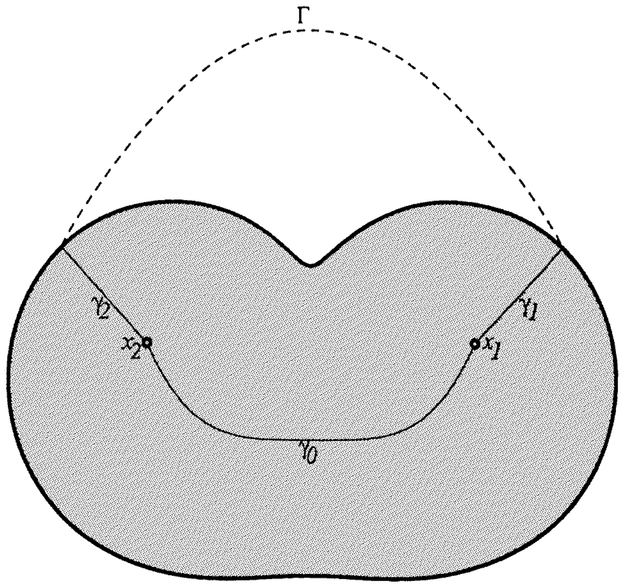

Figure 1. Let γ_ℓ, $\ell = 1, 2$, be a minimal geodesic connecting the *well* $\{x_\ell\}$ with the *ocean* U. The Agmon length of γ_ℓ is S. In the hypercritical case **H** we have that the path given by $\gamma_1 + \Gamma + \gamma_2$, where Γ is any path contained in the *ocean* U and linking the endpoints of γ_1 and γ_2, has Agmon length $2S$. Any local geodesic γ_0 connecting the wells and contained in Ö has length strictly greater than $S_0 = 2S$.

Remark 1. Let us stress that assumption (3.13) is very general and actually implies

$$\overline{B(x_1, S)} \cap \overline{B(x_2, S)} = \emptyset \tag{3.15}$$

in the case **H**. Moreover, let us note that the ocean U is a connected set since (3.2) for $n > 1$; for this reason (using the triangular inequality) we have $S_0 = 2S$ in the hypercritical case and for $n > 1$. The results we give in the next sections for this class of models still hold for one-dimensional models even if the ocean of a one-dimensional double volcano model is not connected.

In the following let us drop the dependence on ν and $\hbar$ where this does not cause misunderstanding and let C denote a generic positive constant.

4. Single volcano resonances

We define resonances in the Helffer–Sjöstrand framework where the potential considered admits real-analytic extension outside of a compact set and satisfies non-trapping conditions. By assuming that the potential V satisfies hypotheses 1–3 given in the appendix we introduce a family of Hilbert spaces $\{\mathcal{H}_s^t\}_s$ (see the appendix) for any $0 < t < t^\star$, where $t^\star > 0$ is fixed and independent of $\hbar$ constant. The multiplicative operator defined as

$$W : \mathcal{H}_2^t \to \mathcal{H}_0^t \tag{4.1}$$

is norm bounded with bound (see lemma A.1 in the appendix)

$$\|W\|_{\mathcal{L}(\mathcal{H}_2^t, \mathcal{H}_0^t)} \leqslant C_1 \mathrm{e}^{C_2 t/\hbar} \tag{4.2}$$

for some constants $C_1 > 0$ and $C_2 \geqslant 0$, where $C_2 = 0$ if $\mathcal{W} \subset \ddot{\mathrm{O}}$. Here we choose

$$t^\star < c\frac{s}{1+C_2} \qquad c = \frac{1}{20} \qquad \text{and} \qquad |\nu| \leqslant \mathrm{e}^{-\tilde{C}t^\star/\overline{h}} \qquad \tilde{C} := 2C_2 + 1. \tag{4.3}$$

The choice of the numerical value of c will be justified in remark 5.

From (4.2) it follows that the resonance operator P_ν^t, formally defined by P_ν from $\mathcal{H}_2^t$ to $\mathcal{H}_0^t$, is well defined and the resonances of P_ν are defined as the eigenvalues of P_ν^t for some $t > 0$ (see [8] and see also theorem 1 and definition 2 in [5]). The strategy to compute the first level resonances of the double well operator P_ν is the same we used in [5]: we start by studying the first level resonance of the two single well operators obtained by filling one well, and then we consider the interaction between the two wells.

Let V_1 be a $C_0^\infty(\mathbb{R}^n, [0, 1])$ function with support contained in $B(x_2, \eta)$ and such that $V(x) + V_1(x) > 0$ for any $x \in B(x_2, \eta)$, where $0 < \eta < s/3$ is fixed and small enough. V_2 is defined as

$$V_2 = \mathcal{S}V_1. \tag{4.4}$$

By construction, the single well operator $P_\ell := P + V_\ell$, $\ell = 1, 2$, admits exactly one single well resonance $z_\ell(\overline{h}) := z_\ell(\overline{h}, \eta)$ close to the ground state $e_1\overline{h}$ of the harmonic oscillator associated, where $e_1 = \sum_{j=1}^n \sqrt{\lambda_j}$ and where $2\lambda_j > 0$ are the eigenvalues of the Hessian matrix of the potential V at the minimum. That is $z_\ell(\overline{h})$ is an eigenvalue of

$$P_\ell^t : \mathcal{H}_2^t \longrightarrow \mathcal{H}_0^t \tag{4.5}$$

for some $0 < t < t^\star$ small enough, with associated eigenfunction $\varphi_\ell(x; \overline{h}) := \varphi_\ell(x; \overline{h}, \eta)$ independent of t.

Remark 2. Since (3.3) and (4.4) it follows that

$$\mathcal{S}P_1 = P_2\mathcal{S} \tag{4.6}$$

and so

$$\tilde{z}(\overline{h}) := z_1(\overline{h}) = z_2(\overline{h}) \qquad \text{and} \qquad \varphi_1 = \mathcal{S}\varphi_2. \tag{4.7}$$

Let us now recall the following properties (see [8] and theorem 7 in [5]).

The imaginary part of the resonance $z_\ell(\overline{h})$ is exponentially small, that is

$$|\Im z_\ell(\overline{h})| = \tilde{\mathcal{O}}(\mathrm{e}^{-2S/\overline{h}}) \qquad \text{as } \overline{h} \to 0 \tag{4.8}$$

where $g = \tilde{\mathcal{O}}(f)$, for two given functions f and g, means that there exists $\epsilon = \epsilon(\eta) > 0$ and a positive constant C_η independent of $\overline{h}$, such that $|g| \leqslant C_\eta \mathrm{e}^{\epsilon/\overline{h}}|f|$ as $\overline{h}$ goes to zero, with $\epsilon(\eta) \to 0$ as $\eta \to 0$.

If the potential is analytic on a neighbourhood of the minimal geodesics connecting the well $\{x_\ell\}$ with the ocean U we have that

$$\Im z_\ell(\overline{h}) = -\phi_\ell(\overline{h}, \eta)\mathrm{e}^{-2S/\overline{h}} \qquad C^{-1}\overline{h}^{1/2} \leqslant \phi_\ell \leqslant C\overline{h}^{1-n/2} \tag{4.9}$$

for some positive C.

The eigenfunction $\varphi_\ell(x; \overline{h})$ associated to $z_\ell(\overline{h})$ can be normalized as

$$(\varphi_\ell, \varphi_\ell) = 1 \tag{4.10}$$

where $(\cdot, \cdot)$ denotes the Euclidean bilinear form

$$(u, v) := \langle u, \bar{v}\rangle_{L^2} = \int u(x)v(x)\,\mathrm{d}x. \tag{4.11}$$

The eigenfunction $\varphi_\ell(x;\hbar)$ satisfies to the following behaviour together with its derivatives:

$$\varphi_\ell(x;\hbar) = \tilde{\mathcal{O}}(\mathrm{e}^{-\rho(x,x_\ell)/\hbar}) \qquad \text{uniformly for } x \in \Omega \text{ as } \hbar \to 0 \tag{4.12}$$

where $\Omega \subset \mathbb{R}^n$ is any bounded open domain. Moreover, let $\mathcal{B}_\ell$ be the union of all null-bicharacteristic curves of $\xi^2 + V(x)$ starting from $\overline{B(x_\ell, S)} \cap \partial\ddot{\mathrm{O}}$; then for any compact set $K \subset \mathbb{R}^n \backslash (\mathcal{B}_\ell \cup \overline{B(x_\ell, S)})$ there exists $\epsilon_0 := \epsilon_0(K) > 0$ such that

$$\varphi_\ell(x;\hbar) = \mathcal{O}(\mathrm{e}^{-(S+\epsilon_0)/\hbar}) \qquad \text{as } \hbar \to 0 \tag{4.13}$$

uniformly for $x \in K$ (see proposition 9.12 of [8]).

By restricting x to the set Ω_ℓ, consisting of x_ℓ and of the interior of the union of all minimal and regular geodesics from x_ℓ to some point of Ö of length strictly less than S, we have that

$$\varphi_\ell(x;\hbar) = \hbar^{-\frac{n}{4}} a_\ell(x;\hbar)\mathrm{e}^{-\rho(x,x_\ell)/\hbar} \qquad x \in \Omega_\ell \tag{4.14}$$

where $a_\ell(x;\hbar)$ is a classical symbol admitting real asymptotic expansion as $\hbar$ goes to zero

$$a_\ell(x;\hbar) \sim a_\ell^0(x) + \hbar a_\ell^1(x) + \cdots \qquad a_\ell^0(x) > 0. \tag{4.15}$$

Now, by using these results and by means of perturbative techniques, we compute the first-level resonance of the single well operator formally defined by

$$P_{\nu,\ell} := P_\ell + \nu W. \tag{4.16}$$

Let

$$J(\delta) := [0, (e_1 + \Lambda)\hbar] \times [-\mathrm{i}\delta, 0] \qquad \delta > 0 \text{ and } \Lambda := \min_j \sqrt{\lambda_j} \tag{4.17}$$

be a box containing the first level resonances $z_\ell(\hbar)$ of P_ℓ and let

$$\widetilde{P}_\nu := P_\nu + V_1 + V_2. \tag{4.18}$$

From the Helffer–Sjöstrand results [8] and the regular perturbation theory it follows that for $0 < t < t^\star$ fixed and small enough the operator

$$P_{\nu,\ell}^t : \mathcal{H}_2^t \to \mathcal{H}_0^t \tag{4.19}$$

has discrete spectrum in a neighbourhood of 0 and so the box $J(\delta)$ is disjoint from the essential spectrum of $P_{\nu,\ell}^t$ for positive δ small enough.

We can state the following theorem:

Theorem 3. For any $0 < t < t^\star$, there exists $\hbar_0 > 0$ and $\delta > 0$ such that for any $\hbar \in (0, \hbar_0]$ and for any $|\nu| \leqslant \mathrm{e}^{-\tilde{C}t^\star/\hbar}$ each operator $P_{\nu,\ell}^t$, $\ell = 1, 2$, has exactly one eigenvalue $z_\ell^\nu(\hbar) := z_\ell^\nu(\hbar, \eta)$ in the box $J(\delta)$ and it is given by:

$$z_\ell^\nu(\hbar) = \tilde{z}(\hbar) + \nu(W\varphi_\ell, \varphi_\ell) + \mathcal{R}(\nu, \hbar) \tag{4.20}$$

where

$$\mathcal{R}(\nu, \hbar) = -\nu^2((\widetilde{P_\nu^t} - \tilde{z}(\hbar))^{-1} W\varphi_\ell, W\varphi_\ell) + \nu\tilde{\mathcal{O}}(\mathrm{e}^{-5\xi/\hbar}) \qquad \text{as } \hbar \to 0 \tag{4.21}$$

and $\xi = \frac{1}{2}s - C_2 t$.

Proof. By construction $\widetilde{P}_0$ has both wells filled and so we have that $(\widetilde{P_0^t} - z)$ is invertible and the inverse operator is uniformly norm bounded for any $z \in J(\delta)$ for $\delta > 0$ small enough (see, for instance, proposition 9.3 and lemma 9.4 in [8]). Then, for ν small enough, $(\widetilde{P_\nu^t} - z)$ is invertible for any z close to $\tilde{z}(\hbar)$ with uniformly bounded inverse operator. Indeed, for some $\delta > 0$

$$(\widetilde{P_\nu^t} - z)^{-1} = (\widetilde{P_0^t} - z)^{-1}\left[1 + \nu W(\widetilde{P_0^t} - z)^{-1}\right]^{-1} \tag{4.22}$$

is uniformly norm bounded with respect to $\hbar$ and $z \in J(\delta)$ for any ν such that $|\nu| \leqslant \mathrm{e}^{-\tilde{C}t^\star/\hbar}$ since (4.2), (4.3) and since $(\widetilde{P_0^t} - z)^{-1}$ is norm bounded. Moreover:

Lemma 4. The kernel of the resolvent of $(\widetilde{P_\nu^t} - z)$ is exponentially decreasing off the diagonal:

$$K_{(\widetilde{P_\nu^t} - z)^{-1}}(x, y) = \tilde{\mathcal{O}}(\mathrm{e}^{-\rho(x,y)/\hbar}) \qquad x, y \in \mathbb{R}^n \qquad \text{as } \hbar \to 0 \tag{4.23}$$

for any $z \in J$.

Proof. If W is a C_0^∞ function with support inside Ö then the result follows from proposition 9.3 and lemma 9.4 of [8]. In any case we have that W is a L^∞ function compactly supported and then we can write

$$(\widetilde{P_\nu^t} - z)^{-1} = (\widetilde{P_0^t} - z)^{-1} \sum_{k=0}^{\infty} (-\nu)^k \left[W(\widetilde{P_0^t} - z)^{-1}\right]^k \tag{4.24}$$

which converges since (4.3). Now, let $\{\psi_{y_j}\}_{j=1}^{N_0}$, $N_0 \in \mathbb{N}$, be a partition of unit adapted to the support of W:

$$W \equiv W\Phi \qquad \Phi := \sum_{j=1}^{N_0} \psi_{y_j} \tag{4.25}$$

and with

$$\operatorname{supp} \psi_{y_j} \subset \{x \in \mathbb{R}^n : |x - y_j| \leqslant \epsilon\} \tag{4.26}$$

where $\epsilon > 0$ is arbitrary. Let $x_0, y_0 \in \mathbb{R}^n$ and let χ_{x_0} and χ_{y_0} be cut-off functions with supports close enough to x_0 and y_0; we have that

$$\begin{aligned}
\|\chi_{x_0}(\widetilde{P_\nu^t} - z)^{-1}\chi_{y_0}\|_{\mathcal{L}(\mathcal{H}_0^t, \mathcal{H}_2^t)} &= \|\chi_{x_0}(\widetilde{P_0^t} - z)^{-1} \sum_{k=0}^{\infty} (-\nu)^k \left[W(\widetilde{P_0^t} - z)^{-1}\right]^k \chi_{y_0}\| \\
&= \|\chi_{x_0}(\widetilde{P_0^t} - z)^{-1} \sum_{k=0}^{\infty} (-\nu)^k \left[\Phi W \Phi(\widetilde{P_0^t} - z)^{-1}\right]^k \chi_{y_0}\| \\
&\leqslant \sum_{k=0}^{\infty} \|W\|^k |\nu|^k \sum_{j,j'=1}^{N_0} \|\chi_{x_0}(\widetilde{P_0^t} - z)^{-1}\psi_{y_j}\| \cdot \|\psi_{y_{j'}}(\widetilde{P_0^t} - z)^{-1}\chi_{y_0}\| \\
&\quad \times \left(\sum_{j,j'=1}^{N_0} \|\psi_{y_j}(\widetilde{P_0^t} - z)^{-1}\psi_{y_{j'}}\| \right)^{k-1}
\end{aligned}$$

and then, using that the result is true for $(\widetilde{P_0^t} - z)^{-1}$ and the norm estimate (4.2) of the perturbation, we get that there exists $\epsilon' = \epsilon'(\epsilon) > 0$, where $\epsilon'(\epsilon) \to 0$ as $\epsilon \to 0$, and a positive constant C_ϵ independent of $\hbar$ such that

$$\|\chi_{x_0}(\widetilde{P_\nu^t} - z)^{-1}\chi_{y_0}\|_{\mathcal{L}(\mathcal{H}_0^t, \mathcal{H}_2^t)} \leqslant C_\epsilon \mathrm{e}^{(\epsilon' - \rho(x_0, y_0))/\hbar} \sum_{k=0}^{\infty} (|\nu| N_0 C_\epsilon \|W\| \mathrm{e}^{\epsilon'/\hbar})^k.$$

The result follows for ϵ and $\hbar$ small enough since $|\nu| \leqslant \mathrm{e}^{-\tilde{C}t^\star/\hbar}$ where $\tilde{C} = 2C_2 + 1 > 0$. □

Let now

$$\mathcal{P}_\ell(z) : \mathcal{H}_2^t \oplus \mathbb{C} \longrightarrow \mathcal{H}_0^t \oplus \mathbb{C} \tag{4.27}$$

be the Grushin type operator acting as

$$\mathcal{P}_\ell(z) = \begin{pmatrix} P_\ell^t - z & \varphi_\ell \\ (\cdot, \varphi_\ell) & 0 \end{pmatrix}. \tag{4.28}$$

It is invertible for z close to $\tilde{z}$ with inverse

$$\mathcal{P}_\ell^{-1}(z) : \mathcal{H}_0^t \oplus \mathbb{C} \longrightarrow \mathcal{H}_2^t \oplus \mathbb{C} \tag{4.29}$$

given by

$$\mathcal{P}_\ell^{-1}(z) = \begin{pmatrix} E_\ell(z) & E_\ell^+ \\ E_\ell^- & E_\ell^{-+}(z) \end{pmatrix} \tag{4.30}$$

where

$$E_\ell(z) = (\widehat{P_\ell^t} - z)^{-1} \qquad E_\ell^+ = \varphi_\ell \qquad E_\ell^- = (\cdot, \varphi_\ell) \qquad E_\ell^{-+}(z) = z - \tilde{z} \tag{4.31}$$

and where $\widehat{P_\ell^t} = (\mathbb{1} - \Pi_\ell) P_\ell^t (\mathbb{1} - \Pi_\ell)$ and Π_ℓ is the spectral projection of P_ℓ^t on the vector φ_ℓ: i.e. $\Pi_\ell \varphi = (\varphi, \varphi_\ell)\varphi_\ell$. Now, let

$$\mathcal{P}_{\nu,\ell}(z) : \mathcal{H}_2^t \oplus \mathbb{C} \longrightarrow \mathcal{H}_0^t \oplus \mathbb{C} \tag{4.32}$$

be the Grushin type operator acting as

$$\mathcal{P}_{\nu,\ell}(z) = \begin{pmatrix} P_{\nu,\ell}^t - z & \varphi_\ell \\ (\cdot, \varphi_\ell) & 0 \end{pmatrix} \tag{4.33}$$

and let

$$\mathcal{E}_{\nu,\ell}(z) : \mathcal{H}_0^t \oplus \mathbb{C} \longrightarrow \mathcal{H}_2^t \oplus \mathbb{C} \tag{4.34}$$

be the operator acting as

$$\mathcal{E}_{\nu,\ell}(z) = \begin{pmatrix} \widetilde{E}_\ell(z) & E_\ell^+ \\ E_\ell^- & E_\ell^{-+}(z) \end{pmatrix} \tag{4.35}$$

where

$$\widetilde{E}_\ell(z) = E_\ell(z)\psi + (\widetilde{P_\nu^t} - z)^{-1}(1 - \psi) \tag{4.36}$$

and $\psi \in C_0^\infty(\mathbb{R}^n, [0, 1])$ is such that $\psi \equiv 1$ on $B(\{x_1, x_2\}, \frac{1}{2}s - \eta)$ and $\psi \equiv 0$ for any $x \notin B(\{x_1, x_2\}, \frac{1}{2}s)$, where η has been previously defined.

Now, we have that

$$\mathcal{P}_{\nu,\ell}(z)\mathcal{E}_{\nu,\ell}(z) = \mathbb{1} + \mathcal{K}_{\nu,\ell}(z) \qquad \mathcal{K}_{\nu,\ell}(z) = \begin{pmatrix} \mathcal{K}_{1,1}(z) & \mathcal{K}_{1,2}(z) \\ \mathcal{K}_{2,1}(z) & \mathcal{K}_{2,2}(z) \end{pmatrix} \tag{4.37}$$

where

$$\mathcal{K}_{1,2}(z) := \mathcal{K}_{1,2} = \nu W \varphi_\ell \qquad \mathcal{K}_{2,2}(z) := \mathcal{K}_{2,2} = 0 \tag{4.38}$$

and where for any test function $u \in \mathcal{H}_0^t$ we have

$$\mathcal{K}_{1,1}(z)u := \mathcal{K}_{1,1}u = \nu W(\widehat{P_\ell^t} - z)^{-1}\psi u - V_{\ell'}(\widetilde{P_\nu^t} - z)^{-1}\theta + \varphi_\ell(\theta, \varphi_\ell)$$

and

$$\begin{aligned} \mathcal{K}_{2,1}(z)u := \mathcal{K}_{2,1}u &= ((\widehat{P_\ell^t} - z)^{-1}\psi u, \varphi_\ell) + ((\widetilde{P_\nu^t} - z)^{-1}\theta, \varphi_\ell) \\ &= ((\widetilde{P_\nu^t} - z)^{-1}\theta, \varphi_\ell) \end{aligned}$$

with $\ell' = 1$ if $\ell = 2$, $\ell' = 2$ if $\ell = 1$, and $\theta := (1-\psi)u$. From (4.12), $\rho(\mathcal{W}, \operatorname{supp}\psi) \geqslant \frac{1}{2}s$, $\rho(\operatorname{supp} V_{\ell'}, \operatorname{supp}(1-\psi)) \geqslant \frac{1}{2}s - 2\eta$ and since the kernels of $(\widetilde{P^t_\nu} - z)^{-1}$ and $(\widehat{P^t_\ell} - z)^{-1}$ are exponentially decreasing outside the diagonal (see lemma 4 and proposition 9.3 and lemma 9.4 of [8]), then we obtain that

$$\|\mathcal{K}_{1,1}\| = \tilde{\mathcal{O}}(\mathrm{e}^{-\xi/\hbar}) \qquad \|\mathcal{K}_{1,2}\| = \tilde{\mathcal{O}}(\mathrm{e}^{-2\xi/\hbar}) \qquad \|\mathcal{K}_{2,1}\| = \tilde{\mathcal{O}}(\mathrm{e}^{-\xi/\hbar}) \tag{4.39}$$

as $\hbar$ goes to zero. In particular, we have

$$\|\mathcal{K}_{\nu,\ell}(z)\| = \tilde{\mathcal{O}}(\mathrm{e}^{-\xi/\hbar}) \qquad \text{as } \hbar \to 0. \tag{4.40}$$

Let us point out that this asymptotic behaviour is uniform for any z in a neighbourhood of $\tilde{z}$ and it holds also for the derivative of $\mathcal{K}_{\nu,\ell}(z)$ with respect to z. From (4.37) and (4.40) it follows that $\mathcal{P}_{\nu,\ell}(z)$ admits a right inverse given by:

$$\mathcal{P}^{-1}_{\nu,\ell}(z) = \begin{pmatrix} E_{\nu,\ell}(z) & E^+_{\nu,\ell} \\ E^-_{\nu,\ell} & E^{-+}_{\nu,\ell}(z) \end{pmatrix} = \mathcal{E}_{\nu,\ell}(z) \sum_{k=0}^{\infty} \left[-\mathcal{K}_{\nu,\ell}(z)\right]^k \tag{4.41}$$

and in the same way the existence of the left inverse follows. By construction we also have that $z \in \sigma(P^t_{\nu,\ell})$ if and only if $E^{-+}_{\nu,\ell}(z) = 0$. Therefore, it remains to compute $E^{-+}_{\nu,\ell}(z)$. In order to do this we have that as $\hbar$ goes to zero:

$$\begin{aligned}
E^{-+}_{\nu,\ell}(z) = {} & E^{-+}_\ell(z) - E^-_\ell\mathcal{K}_{1,2} + E^-_\ell\mathcal{K}_{1,1}\mathcal{K}_{1,2} + E^{-+}_\ell(z)\mathcal{K}_{2,1}\mathcal{K}_{1,2} \\
& - E^-_\ell(\mathcal{K}^2_{1,1}\mathcal{K}_{1,2} + \mathcal{K}_{1,2}\mathcal{K}_{2,1}\mathcal{K}_{1,2}) + E^{-+}_\ell(z)\mathcal{K}_{2,1}\mathcal{K}_{1,1}\mathcal{K}_{1,2} \\
& + E^-_\ell(\mathcal{K}^3_{1,1}\mathcal{K}_{1,2} + \mathcal{K}_{1,2}\mathcal{K}_{2,1}\mathcal{K}_{1,1}\mathcal{K}_{1,2} + \mathcal{K}_{1,1}\mathcal{K}_{1,2}\mathcal{K}_{2,1}\mathcal{K}_{1,2}) \\
& + E^{-+}_\ell(z)(\mathcal{K}_{2,1}\mathcal{K}^2_{1,1}\mathcal{K}_{1,2} + \mathcal{K}_{2,1}\mathcal{K}_{1,2}\mathcal{K}_{2,1}\mathcal{K}_{1,2}) + \mathcal{O}(\|\mathcal{K}_{1,2}\| \cdot \|\mathcal{K}_{\nu,\ell}(z)\|^4) \\
= {} & E^{-+}_\ell(z) - E^-_\ell\mathcal{K}_{1,2} + E^-_\ell\mathcal{K}_{1,1}\mathcal{K}_{1,2} + E^{-+}_\ell(z)\mathcal{K}_{2,1}\mathcal{K}_{1,2} \\
& - E^-_\ell\mathcal{K}^2_{1,1}\mathcal{K}_{1,2} + E^{-+}_\ell(z)\mathcal{K}_{2,1}\mathcal{K}_{1,1}\mathcal{K}_{1,2} + \tilde{\mathcal{O}}(\nu\mathrm{e}^{-5\xi/\hbar}) \\
= {} & z - \tilde{z} - \nu(V_{\ell'}(\widetilde{P^t_\nu} - z)^{-1}W\varphi_\ell, \varphi_\ell) + \nu(z-\tilde{z})((\widetilde{P^t_\nu} - z)^{-1}W\varphi_\ell, \varphi_\ell) \\
& + \tilde{\mathcal{O}}(\nu\mathrm{e}^{-5\xi/\hbar}).
\end{aligned}$$

Indeed:

$$E^-_\ell\mathcal{K}_{1,2} = (\mathcal{K}_{1,2}, \varphi_\ell) = \nu(W\varphi_\ell, \varphi_\ell)$$

$$\begin{aligned}
E^-_\ell\mathcal{K}_{1,1}\mathcal{K}_{1,2} &= (\mathcal{K}_{1,1}\mathcal{K}_{1,2}, \varphi_\ell) \\
&= -\nu[(V_{\ell'}(\widetilde{P^t_\nu} - z)^{-1}W\varphi_\ell, \varphi_\ell) - (W\varphi_\ell, \varphi_\ell)]
\end{aligned}$$

$$E^{-+}_\ell\mathcal{K}_{2,1}\mathcal{K}_{1,2} = \nu(z-\tilde{z})((\widetilde{P^t_\nu} - z)^{-1}W\varphi_\ell, \varphi_\ell)$$

$$\begin{aligned}
E^-_\ell\mathcal{K}^2_{1,1}\mathcal{K}_{1,2} &= (\mathcal{K}^2_{1,1}\mathcal{K}_{1,2}, \varphi_\ell) = \nu(\mathcal{K}^2_{1,1}W\varphi_\ell, \varphi_\ell) \\
&= -\nu[(\mathcal{K}_{1,1}V_{\ell'}(\widetilde{P^t_\nu} - z)^{-1}W\varphi_\ell, \varphi_\ell) - (\mathcal{K}_{1,1}\varphi_\ell, \varphi_\ell)(W\varphi_\ell, \varphi_\ell)] \\
&= -\nu[\nu((\widehat{P^t_\nu} - z)^{-1}V_{\ell'}(\widetilde{P^t_\nu} - z)^{-1}W\varphi_\ell, W\varphi_\ell) + \tilde{\mathcal{O}}(\mathrm{e}^{-5\xi/\hbar})] \\
&= \nu[\tilde{\mathcal{O}}(\mathrm{e}^{-6\xi/\hbar}) + \tilde{\mathcal{O}}(\mathrm{e}^{-5\xi/\hbar})] = \nu\tilde{\mathcal{O}}(\mathrm{e}^{-5\xi/\hbar})
\end{aligned}$$

and

$$\begin{aligned}
E^{-+}\mathcal{K}_{2,1}\mathcal{K}_{1,1}\mathcal{K}_{1,2} &= \nu(z-\tilde{z})((\widetilde{P^t_\nu} - z)^{-1}(1-\psi)\mathcal{K}_{1,1}W\varphi_\ell, \varphi_\ell) \\
&= \nu(z-\tilde{z})((\widetilde{P^t_\nu} - z)^{-1}(1-\psi)\varphi_\ell, \varphi_\ell)(W\varphi_\ell, \varphi_\ell) \\
&= \nu\tilde{\mathcal{O}}(\mathrm{e}^{-5\xi/\hbar})
\end{aligned}$$

since $\psi W \equiv 0$, $W\varphi_\ell = \tilde{\mathcal{O}}(\mathrm{e}^{-2\xi/\overline{h}})$, $V_{\ell'}(\widetilde{P_\nu^t} - z)^{-1}W = \tilde{\mathcal{O}}(\mathrm{e}^{-2\xi/\overline{h}})$ and $(1-\psi)\varphi_\ell = \tilde{\mathcal{O}}(\mathrm{e}^{-s/2\overline{h}})$. Moreover, the second resolvent formula gives

$$(z - \tilde{z})((\widetilde{P_\nu^t} - z)^{-1}W\varphi_\ell, \varphi_\ell) = (W\varphi_\ell, -\varphi_\ell + (\widetilde{P_\nu^t} - z)^{-1}(\nu W + V_{\ell'})\varphi_\ell)$$

and so we obtain as $\overline{h}$ goes to zero:

$$E_{\nu,\ell}^{-+}(z) = z - \tilde{z} - \nu\big[(W\varphi_\ell, \varphi_\ell) - \nu\big(\big(\widetilde{P_\nu^t} - z\big)^{-1}W\varphi_\ell, W\varphi_\ell\big) - \tilde{\mathcal{O}}\big(\mathrm{e}^{-5\xi/\overline{h}}\big)\big].$$

Therefore, the solution of $E_{\nu,\ell}^{-+}(z) = 0$ is given by the fixed point theorem:

$$z_\ell^\nu = \tilde{z} + \nu[(W\varphi_\ell, \varphi_\ell) - \nu((\widetilde{P_\nu^t} - \tilde{z})^{-1}W\varphi_\ell, W\varphi_\ell) + \tilde{\mathcal{O}}(\mathrm{e}^{-5\xi/\overline{h}})]$$

so proving the theorem. □

Remark 5. If $\mathcal{W} \subset \mathring{\mathrm{O}}$ then we have that $C_2 = 0$ and so $\xi = \frac{1}{2}s$. Thus the remainder becomes

$$\mathcal{R}(\nu, \overline{h}) = -\nu^2((\widetilde{P_\nu^t} - \tilde{z}(\overline{h}))^{-1}W\varphi_\ell, W\varphi_\ell) + \nu\tilde{\mathcal{O}}(\mathrm{e}^{-5s/2\overline{h}}). \tag{4.42}$$

In contrast, if $\mathcal{W} \cap U \neq \emptyset$, then $C_2 \neq 0$ in general and so the remainder becomes

$$\mathcal{R}(\nu, \overline{h}) = -\nu^2((\widetilde{P_\nu^t} - \tilde{z}(\overline{h}))^{-1}W\varphi_\ell, W\varphi_\ell) + \nu\tilde{\mathcal{O}}(\mathrm{e}^{-9s/4\overline{h}}) \tag{4.43}$$

since (4.3) where $c = \frac{1}{20}$.

Remark 6. Let $\varphi_\ell^\nu := \varphi_\ell^\nu(x; \overline{h})$, $\ell = 1, 2$, be the eigenvector of $P_{\nu,\ell}^t$ associated to $z_\ell^\nu(\overline{h})$. Then, it can be normalized in the sense that $(\varphi_\ell^\nu, \varphi_\ell^\nu) = 1$. Moreover, when the perturbation W is given by a C_0^∞ function, it satisfies to the asymptotic behaviours (4.12), (4.13) and (4.14) already given for φ_ℓ because of the results of Helffer and Sjöstrand [8] applied to P_ℓ^ν and where the Agmon distance does not change, i.e. $\rho := \rho_V$ again, since the perturbative parameter ν is exponentially small in $\overline{h}$. As for the external perturbation case, where the perturbation W is given by a non-smooth function, we have again similar asymptotic behaviours because the support of the perturbation is a compact set contained in the ocean U. More precisely, by using a perturbative argument, we have that for any compact set K contained in $\dot{\mathrm{O}}\backslash\overline{B(x_\ell, S)}$ there exists $\epsilon_0 := \epsilon_0(K) > 0$ such that

$$\varphi_\ell^\nu(x; \overline{h}) = \mathcal{O}(\mathrm{e}^{-(S+\epsilon_0)/\overline{h}}) \qquad \text{as } \overline{h} \to 0 \tag{4.44}$$

uniformly for any $x \in K$ and for any $|\nu| \leqslant \mathrm{e}^{-\tilde{C}t^\star/\overline{h}}$. Indeed, we have that φ_ν^ℓ is given, up to a normalization constant, by

$$\begin{aligned}\varphi_\ell^\nu &= -\frac{1}{2\pi i}\oint_{\partial J}(P_{\nu,\ell}^t - z)^{-1}\varphi_\ell\,\mathrm{d}z \\ &= \varphi_\ell + \sum_{k=1}^\infty \frac{(-\nu)^k}{2\pi\mathrm{i}}\oint_{\partial J}(P_\ell^t - z)^{-1}[W(P_\ell^t - z)^{-1}]^k\varphi_\ell\,\mathrm{d}z\end{aligned}$$

and so (4.44) follows by using that the result is true for $\nu = 0$ and proving, as in the proof of lemma 4, that the remainder terms can be estimated by $\mathcal{O}(\mathrm{e}^{-(S+\epsilon_0)/\overline{h}})$ uniformly on K for some $\epsilon_0 > 0$ as $\overline{h}$ goes to zero. In the same way (4.12) and (4.14) follow too.

5. Double volcano resonances.

In this section we consider the interaction between the two wells. As in section 4 we have that for any $0 < t < t^\star$ small enough the operator

$$P_\nu^t : \mathcal{H}_2^t \longrightarrow \mathcal{H}_0^t \tag{5.1}$$

has discrete spectrum in a neighbourhood of 0 and so the box $J := J(\delta)$ is disjoint from the essential spectrum of P_ν^t for some positive δ small enough. Let

$$\Pi_\nu^t = -\frac{1}{2\pi \mathrm{i}} \oint_{\partial J} (P_\nu^t - z)^{-1} \, \mathrm{d}z \qquad \Pi_\nu^t : \mathcal{H}_0^t \longrightarrow \mathcal{H}_2^t \tag{5.2}$$

be the spectral eigenprojection of P_ν^t on J.

We have that $\operatorname{Ran} \Pi_\nu^t = 2$ and then the double well operator P_ν^t has exactly two eigenvalues $E_1^\nu(\overline{h})$ and $E_2^\nu(\overline{h})$ in the box J for $\overline{h}$ small enough. That is (see theorem 9 in [5]):

Theorem 7. For any $0 < t < t^\star$ small enough, there exists $\overline{h}_0 > 0$ and $\delta > 0$ such that for any $\overline{h} \in (0, \overline{h}_0]$ and for any $|\nu| \leqslant \mathrm{e}^{-\tilde{C} t^\star/\overline{h}}$, the operator P_ν^t admits exactly two eigenvalues $E_\ell^\nu := E_\ell^\nu(\overline{h})$, $\ell = 1, 2$, in the box $J(\delta)$.

Now, we are going to perform the asymptotic evaluation of these two eigenvalues and of the associated eigenvectors $\Phi_\ell^\nu(x; \overline{h})$. Besides, we give a criterion of localization where:

Definition. Let $\mathbb{1}_{B(x_\ell, r)}$ be the characteristic function on a fixed neighbourhood $B(x_\ell, r)$ of the well $\{x_\ell\}$ where $r > 0$ is fixed and small enough. The resonant state associated to $E_\ell^\nu(\overline{h})$ is delocalized (on both wells) if there exist two positive constants $\rho := \rho(r)$ and $C := C(r)$ independent of $\overline{h}$ such that:

$$\left| \|\mathbb{1}_{B(x_j, r)} \Phi_\ell^\nu\|_{\mathcal{H}_0^t} - \frac{1}{\sqrt{2}} \right| \leqslant C \mathrm{e}^{-\rho/\overline{h}} \qquad j = 1, 2 \tag{5.3}$$

for $\overline{h}$ small enough. The resonant state associated to $E_\ell^\nu(\overline{h})$ is localized (on the well $\{x_\ell\}$) if there exist two positive constants $\rho := \rho(r)$ and $C := C(r)$ independent of $\overline{h}$ such that:

$$\left| \|\mathbb{1}_{B(x_j, r)} \Phi_\ell^\nu\|_{\mathcal{H}_0^t} - \delta_\ell^j \right| \leqslant C \mathrm{e}^{-\rho/\overline{h}} \qquad j = 1, 2 \tag{5.4}$$

for $\overline{h}$ small enough.

That is, we have localization when each resonant state is asymptotically *concentrate* in a small neighbourhood of one well. In contrast, we have delocalization when each resonant state is asymptotically *concentrate* in a small neighbourhood of both wells.

Let

$$v_\ell^\nu := \Pi_\nu^t \chi_\ell \varphi_\ell^\nu \tag{5.5}$$

be two vectors of $\operatorname{Ran} \Pi_\nu^t$ where χ_ℓ are $C_0^\infty(\mathbb{R}^n, [0, 1])$ functions such that

$$\chi_2 = \mathcal{S} \chi_1 \tag{5.6}$$

and where χ_1 is defined in the following ways:

S *sub-critical case*: $\chi_1 \equiv 0$ on $B(x_2, \eta)$ and $\chi_1 \equiv 1$ outside $B(x_2, 2\eta)$;

H *hypercritical case*: $\chi_1 \equiv 0$ on $B(x_2, S - 2\eta)$ and $\chi_1 \equiv 1$ outside $B(x_2, S - \eta)$;

where $\eta > 0$ has been previously defined. Thus, $\chi_\ell V_\ell \equiv 0$. The different choices for the two cases will be justified in lemmas 8 and 11 below.

Let us stress that in the hypercritical case **H** there exists a fixed $\epsilon_0 > 0$ such that

$$\varphi_1^\nu(x; \overline{h}) = \mathcal{O}(\mathrm{e}^{-(S+\epsilon_0)/\overline{h}}) \qquad \text{as } \overline{h} \to 0 \tag{5.7}$$

uniformly on the closure of $B(x_2, S)$ since (3.13) and remark 6. The same behaviour holds for φ_2^ν uniformly on the closure of $B(x_1, S)$.

Now, we have that the two vectors v_ℓ^ν form a basis of $\mathrm{Ran}\Pi_\nu^t$; in fact:

Lemma 8. Let

$$c_\ell^\nu := (v_\ell^\nu, v_\ell^\nu) \qquad \ell = 1, 2 \tag{5.8}$$

and

$$d^\nu := (v_1^\nu, v_2^\nu) = (v_2^\nu, v_1^\nu). \tag{5.9}$$

In the sub-critical case **S** we have that:

$$c_\ell^\nu = 1 + \tilde{\mathcal{O}}(\mathrm{e}^{-2S_0/\hbar}) \qquad d^\nu = \tilde{\mathcal{O}}(\mathrm{e}^{-S_0/\hbar}) \qquad \text{as } \hbar \to 0. \tag{5.10}$$

In the hypercritical case **H** there exists $\epsilon_0 > 0$ such that

$$c_\ell^\nu = 1 + \mathcal{O}(\mathrm{e}^{-(S_0+2\epsilon_0)/\hbar}) \qquad d^\nu = \mathcal{O}(\mathrm{e}^{-(S_0+2\epsilon_0)/2\hbar}) \qquad \text{as } \hbar \to 0. \tag{5.11}$$

Proof. We can write:

$$v_\ell^\nu = \Pi_\nu^t \chi_\ell \varphi_\ell^\nu = \chi_\ell \varphi_\ell^\nu + R_\ell^\nu \tag{5.12}$$

where R_ℓ^ν is the vector belonging to $\mathcal{H}_0^t$ defined as

$$R_\ell^\nu = \frac{-\hbar^2}{2\pi i} \oint_{\partial J} (z_\ell^\nu - z)^{-1} (P_\nu^t - z)^{-1} [\Delta, \chi_\ell] \varphi_\ell^\nu \, \mathrm{d}z. \tag{5.13}$$

In the sub-critical case **S** we have

$$\|[\Delta, \chi_\ell]\varphi_\ell^\nu\|_{\mathcal{H}_0^t} = \tilde{\mathcal{O}}(\mathrm{e}^{-S_0/\hbar}) \qquad \|R_\ell^\nu\|_{\mathcal{H}_0^t} = \tilde{\mathcal{O}}(\mathrm{e}^{-S_0/\hbar}) \tag{5.14}$$

as $\hbar$ goes to zero since (4.12) and the definition of χ_ℓ. In the hypercritical case **H** we have the different behaviour as $\hbar$ goes to zero:

$$\|[\Delta, \chi_\ell]\varphi_\ell^\nu\|_{\mathcal{H}_0^t} = \mathcal{O}(\mathrm{e}^{-(S_0+2\epsilon_0)/2\hbar}) \qquad \|R_\ell^\nu\|_{\mathcal{H}_0^t} = \mathcal{O}(\mathrm{e}^{-(S_0+2\epsilon_0)/2\hbar}) \tag{5.15}$$

for some $\epsilon_0 > 0$ since (5.7), the definition of χ_ℓ and $2S = S_0$. From now on the proof simply follows the one of lemma 11 in [5]. □

Remark 9. From the asymptotic behaviours (5.10) and (5.11) it follows that the two vectors v_ℓ^ν are linearly independent and so they form a basis of $\mathrm{Ran}\,\Pi_\nu^t$. Such a basis can be put in orthonormal form $\{u_1^\nu, u_2^\nu\}$ (in the sense that $(u_\ell^\nu, u_j^\nu) = \delta_j^\ell$) where:

$$\begin{pmatrix} u_1^\nu \\ u_2^\nu \end{pmatrix} = \mathcal{A}^\nu \begin{pmatrix} v_1^\nu \\ v_2^\nu \end{pmatrix}. \tag{5.16}$$

Since lemma 8, the matrix $\mathcal{A}^\nu$ is such that:

$$\mathcal{A}^\nu = \begin{pmatrix} 1 + \tilde{\mathcal{O}}(\mathrm{e}^{-2S_0/\hbar}) & \tilde{\mathcal{O}}(\mathrm{e}^{-S_0/\hbar}) \\ \tilde{\mathcal{O}}(\mathrm{e}^{-S_0/\hbar}) & 1 + \tilde{\mathcal{O}}(\mathrm{e}^{-2S_0/\hbar}) \end{pmatrix} \qquad \text{as } \hbar \to 0$$

in the sub-critical case **S** and

$$\mathcal{A}^\nu = \begin{pmatrix} 1 + \mathcal{O}(\mathrm{e}^{-(S_0+2\epsilon_0)/\hbar}) & \mathcal{O}(\mathrm{e}^{-(S_0+2\epsilon_0)/2\hbar}) \\ \mathcal{O}(\mathrm{e}^{-(S_0+2\epsilon_0)/2\hbar}) & 1 + \mathcal{O}(\mathrm{e}^{-(S_0+2\epsilon_0)/\hbar}) \end{pmatrix} \qquad \text{as } \hbar \to 0$$

in the hypercritical case **H**. By construction, we have that:

$$u = (u, u_1^\nu) u_1^\nu + (u, u_2^\nu) u_2^\nu \qquad \forall u \in \mathrm{Ran}\,\Pi_\nu^t. \tag{5.17}$$

Now, we are ready to explicitly compute the two resonances $E_1^\nu(\hbar)$ and $E_2^\nu(\hbar)$ as the eigenvalues of the restriction of the operator P_ν^t on Ran Π_ν^t. That is, they are the eigenvalues of the matrix

$$\mathcal{M}_\nu := \begin{pmatrix} \alpha_1^\nu & \beta^\nu \\ \beta^\nu & \alpha_2^\nu \end{pmatrix} \tag{5.18}$$

where

$$\alpha_\ell^\nu := (P_\nu^t u_\ell^\nu, u_\ell^\nu) \tag{5.19}$$

and

$$\beta^\nu := (P_\nu^t u_1^\nu, u_2^\nu) = (P_\nu^t u_2^\nu, u_1^\nu). \tag{5.20}$$

Hence:

$$E_\ell^\nu(\hbar) = \tfrac{1}{2}(\alpha_1^\nu + \alpha_2^\nu) - (-1)^\ell \beta^\nu \sqrt{1+(p^\nu)^2} \qquad p^\nu = \frac{\alpha_1^\nu - \alpha_2^\nu}{2\beta^\nu}. \tag{5.21}$$

The orthonormal eigenvectors of the matrix $\mathcal{M}_\nu$ are given by

$$w_\ell^\nu = \frac{1}{\sqrt{1+(q_\ell^\nu)^2}} \begin{pmatrix} 1 \\ q_\ell^\nu \end{pmatrix} \qquad q_\ell^\nu = -p^\nu + (-1)^\ell \sqrt{1+(p^\nu)^2} \tag{5.22}$$

and then the eigenvectors $\Phi_\ell^\nu(x;\hbar)$ of P_ν^t associated to $E_\ell^\nu(\hbar)$ are given by

$$\Phi_\ell^\nu = \frac{\mathcal{A}_{1,1}^\nu + q_\ell^\nu \mathcal{A}_{2,1}^\nu}{\sqrt{1+(q_\ell^\nu)^2}} v_1^\nu + \frac{\mathcal{A}_{1,2}^\nu + q_\ell^\nu \mathcal{A}_{2,2}^\nu}{\sqrt{1+(q_\ell^\nu)^2}} v_2^\nu. \tag{5.23}$$

Remark 10. Let us stress that we have delocalization in the sense of the above definition when p^ν exponentially goes to zero as $\hbar \to 0$, while we have localization when p^ν exponentially goes to infinity as $\hbar \to 0$. In fact, v_ℓ^ν is localized, up to an exponentially small correction as $\hbar$ goes to zero, on a neighbourhood of the well $\{x_\ell\}$ since (5.12), (5.14), (5.15) and remark 6. Moreover, when $p := p^\nu$ goes to zero we have that

$$w_\ell^\nu = \frac{1}{\sqrt{2}+\mathcal{O}(p^2)} \begin{pmatrix} 1 \\ (-1)^\ell + \mathcal{O}(p) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{(-1)^\ell}{\sqrt{2}} \end{pmatrix} (1+\mathcal{O}(p)). \tag{5.24}$$

While, when $|p|$ goes to infinity we have that

$$w_1^\nu = \frac{1}{1+\mathcal{O}(p^{-2})} \begin{pmatrix} 1 \\ -\frac{1}{2}p^{-1} + \mathcal{O}(p^{-3}) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1+\mathcal{O}(p^{-1})) \tag{5.25}$$

and

$$w_2^\nu = \frac{1}{2p+\mathcal{O}(p^{-1})} \begin{pmatrix} 1 \\ 2p + \mathcal{O}(p^{-1}) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1+\mathcal{O}(p^{-1})). \tag{5.26}$$

Now, we compute the asymptotic behaviour of the elements of $\mathcal{M}_\nu$:

Lemma 11. Let α_ℓ^ν and β^ν as above. In the sub-critical case **S** we have

$$\alpha_\ell^\nu = z_\ell^\nu + \tilde{\mathcal{O}}(\mathrm{e}^{-2S_0/\hbar}) \qquad \beta^\nu = \mu^\nu(\hbar)\mathrm{e}^{-S_0/\hbar} + \tilde{\mathcal{O}}(\mathrm{e}^{-2S_0/\hbar}) \tag{5.27}$$

as $\hbar$ goes to zero, where $\mu^\nu(\hbar)$ is real and such that

$$C^{-1}\hbar^{1/2} \leqslant |\mu^\nu(\hbar)| \leqslant C\hbar^{1-n/2} \tag{5.28}$$

for some positive constant C. In the hypercritical case **H** there exists $\epsilon_0 > 0$ such that

$$\alpha_\ell^\nu = z_\ell^\nu + \mathcal{O}(\mathrm{e}^{-(S_0+2\epsilon_0)/\hbar}) \qquad \beta^\nu = \mathcal{O}(\mathrm{e}^{-(S_0+\epsilon_0/2)/\hbar}) \tag{5.29}$$

as $\hbar$ goes to zero.

Proof. The proof of this lemma essentially follows the one of lemma 13 in [5]; so we briefly show the principal steps dropping out the details. In order to prove the asymptotic behaviour of α_ℓ^ν let us fix ℓ, say $\ell = 1$, and we compute

$$P_\nu^t u_1^\nu = z_1^\nu u_1^\nu + (z_2^\nu - z_1^\nu)\mathcal{A}_{1,2}^\nu v_2^\nu + r_1^\nu \tag{5.30}$$

where

$$r_1^\nu := \sum_{j=1}^{2} \mathcal{A}_{1,j}^\nu \{-\hbar^2[\Delta, \chi_j]\varphi_j^\nu + (P_\nu^t - z_j^\nu)R_j^\nu\}. \tag{5.31}$$

Therefore

$$\alpha_1^\nu = (P_\nu^t u_1^\nu, u_1^\nu) = z_1^\nu + \tilde{\mathcal{O}}(\mathrm{e}^{-2S_0/\hbar}) \qquad \text{as } \hbar \to 0 \tag{5.32}$$

in the sub-critical case **S** since

$$(u_1^\nu, v_2^\nu) = \tilde{\mathcal{O}}(\mathrm{e}^{-S_0/\hbar}) \qquad \text{and} \qquad (u_1^\nu, r_1^\nu) = \tilde{\mathcal{O}}(\mathrm{e}^{-2S_0/\hbar}) \tag{5.33}$$

as $\hbar$ goes to zero from lemma 8 and remark 9. In full analogy, we obtain the asymptotic behaviour of α_ℓ^ν in the hypercritical case **H**. As for the proof of the asymptotic behaviour of β^ν in the sub-critical case **S** we have that as $\hbar$ goes to zero (see again the proof of lemma 13 in [5]):

$$\beta^\nu = \hbar^2 \oint_{\partial B(x_2,2\eta)} [\varphi_1^\nu \nabla\varphi_2^\nu - \varphi_2^\nu \nabla\varphi_1^\nu] \cdot \boldsymbol{n}\,\mathrm{d}\Gamma + \tilde{\mathcal{O}}(\mathrm{e}^{-2S_0/\hbar}) \tag{5.34}$$

where $\boldsymbol{n}$ is the unit interior normal on $\partial B(x_2, 2\eta)$. Since the minimal geodesics connecting the wells are internal in Ö in the sub-critical case **S**, then this integral can be asymptotically evaluated using (4.14) and obtaining thus (5.27). In the hypercritical case **H** we have that (5.34) is replaced by

$$\beta^\nu = \hbar^2 \oint_{\partial B(x_2,S-\eta)} [\varphi_1^\nu \nabla\varphi_2^\nu - \varphi_2^\nu \nabla\varphi_1^\nu] \cdot \boldsymbol{n}\,\mathrm{d}\Gamma + \mathcal{O}(\mathrm{e}^{-(S_0+\epsilon_0/2)/\hbar}). \tag{5.35}$$

and so, since (4.14) and (5.7), the above behaviour of β^ν follows. □

Now, we are ready to give a criterion of localization for the resonant state associated to $E_\ell^\nu(\hbar)$, $\ell = 1, 2$, where $\hbar$ and ν satisfy the conditions in theorem 7.

5.1. Symmetrical double volcano

We consider the semiclassical Schrödinger operator (3.1) with a symmetrical double volcano potential. In such a case the above construction gives

$$v_2 = \mathcal{S} v_1 \tag{5.36}$$

and

$$c := c_1 = c_2 \tag{5.37}$$

because $\mathcal{S}$ is a symmetrical operator such that $\mathcal{S}^2 = \mathbb{1}$ and commutes with complex conjugation. Hence, u_1 and u_2 are given by

$$u_1 = \kappa v_1 + \zeta v_2 \qquad \text{and} \qquad u_2 = \zeta v_1 + \kappa v_2 \tag{5.38}$$

where κ and ζ have the following behaviour in the sub-critical case **S**

$$\kappa = \frac{c + \sqrt{c^2 - d^2}}{2(c^2 - d^2)} = 1 + \tilde{\mathcal{O}}(\mathrm{e}^{-2S_0/\hbar}) \qquad \text{as } \hbar \to 0 \tag{5.39}$$

and

$$\zeta = -\frac{d}{c + \sqrt{c^2 - d^2}} = \tilde{\mathcal{O}}(\mathrm{e}^{-S_0/\overline{h}}) \qquad \text{as } \overline{h} \to 0. \tag{5.40}$$

One can easily check that they are orthonormal and that

$$u_2 = \mathcal{S} u_1 \tag{5.41}$$

since (5.36). Hence, $\alpha := \alpha_1 = \alpha_2$ and so the two resonances of P in J are given by

$$E_{1,2}(\overline{h}) = \alpha \pm \beta. \tag{5.42}$$

Thus, the splitting between the resonances is given by

$$E_2(\overline{h}) - E_1(\overline{h}) = 2\beta = 2\mu(\overline{h})\mathrm{e}^{-S_0/\overline{h}} + \tilde{\mathcal{O}}(\mathrm{e}^{-2S_0/\overline{h}}) \qquad \text{as } \overline{h} \to 0 \tag{5.43}$$

and it is essentially real since lemma 11.

The associated eigenvectors of $\mathcal{M}$ are given by

$$w_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \qquad w_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{5.44}$$

and so we have delocalization like for the stable double well model (see [section 4.3.5, 7]). Indeed, one can check that the eigenvectors of P^t associated to $E_\ell(\overline{h})$ are given by

$$\begin{aligned}\Phi_\ell(x;\overline{h}) &= \frac{1}{\sqrt{2}}(\kappa + (-1)^\ell \zeta)v_1 + \frac{1}{\sqrt{2}}(\zeta + (-1)^\ell \kappa)v_2 \\ &= \frac{1}{\sqrt{2}}[\chi_1\varphi_1 + (-1)^\ell \chi_2\varphi_2] + \tilde{\mathcal{O}}(\mathrm{e}^{-S_0/\overline{h}}) \qquad \text{as } \overline{h} \to 0.\end{aligned}$$

The same delocalization result holds for the hypercritical case **H** too.

5.2. Symmetrical double volcano with internal perturbation

We consider now the semiclassical Schrödinger operator P_ν where the bounded perturbation W which causes the breakdown of the symmetry is *internal* (see figure 2). That is, let s_ℓ be the Agmon distance between the well $\{x_\ell\}$ and the support of W; $s_\ell > 0$ because the support of W does not intersect the wells. We assume that the perturbation is actually asymmetric in the sense that

$$s_1 < s_2 \tag{5.45}$$

and it is *internal* in the sense that

$$s_1 < S. \tag{5.46}$$

Let $A_r := B(x_1, s_1 + r) \cap \mathcal{W}$, where $r > 0$ is such that

$$\mathrm{int}(A_r) \neq \emptyset. \tag{5.47}$$

We can state the following:

Theorem 12. Let W be a bounded internal perturbation, let $\overline{h} \in (0, \overline{h}_0]$ with $\overline{h}_0 > 0$ small enough and let $|\nu| \leqslant \mathrm{e}^{-\tilde{C}t^\star/\overline{h}}$. If $s_1 > \frac{1}{2}S_0$ then the resonant state of P_ν associated to $E_\ell^\nu(\overline{h})$, $\ell = 1, 2$, is delocalized. If $s_1 < \frac{1}{2}S_0$, if ν is such that $-\overline{h}\ln|\nu| < a$, where $a < S_0 - 2s_1$ is independent of $\overline{h}$, and if there exists $r > 0$ such that $A_r \subset \Omega_1$, then the resonant state of P_ν associated to $E_\ell^\nu(\overline{h})$ is localized.

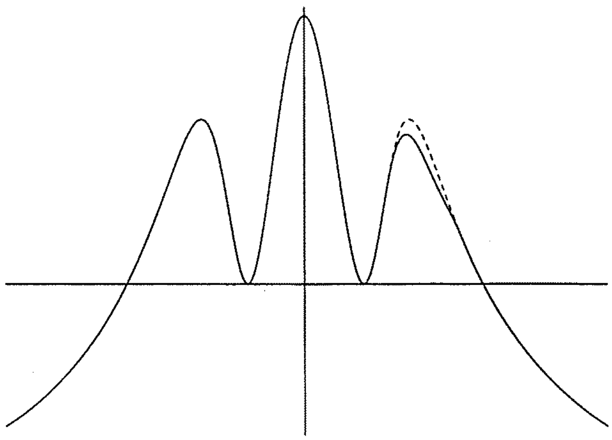

Figure 2. Graph of a one-dimensional double volcano potential with internal perturbation (broken curve denotes the symmetric unperturbed double volcano).

Proof. In order to prove the delocalization result when $s_1 > \frac{1}{2}S_0$ (which is possible only in the sub-critical case **S**) we stress that (4.12), remark 5 (where $s = s_1$), remark 10 and lemma 11 imply, as $\hbar$ goes to zero, that

$$(W\varphi_\ell, \varphi_\ell) = \tilde{\mathcal{O}}(e^{-2s_\ell/\hbar}) \qquad \ell = 1, 2 \tag{5.48}$$

and

$$|\nu((\widetilde{P_\nu^t} - \tilde{z})^{-1} W\varphi_\ell, W\varphi_\ell)| = |\nu| \tilde{\mathcal{O}}(e^{-2(s_\ell - C_2 t)/\hbar}) = \mathcal{O}(e^{-2(s_1+\lambda)/\hbar}) \tag{5.49}$$

for some $\lambda > 0$ since $|\nu| \leqslant e^{-\tilde{C}t^\star/\hbar}$. Thus

$$p^\nu = \nu \frac{\tilde{\mathcal{O}}(e^{-(2s_1 - S_0)/\hbar}) + \tilde{\mathcal{O}}(e^{-S_0/\hbar})}{2\mu^\nu(\hbar)} \qquad \text{as } \hbar \to 0 \tag{5.50}$$

which goes to zero as $\hbar \to 0$ since $s_1 > \frac{1}{2}S_0$, so proving the delocalization result.

In order to prove the localization result in the case $s_1 < \frac{1}{2}S_0$ let us stress that (4.14) holds for any $x \in A_r$ since $A_r \subset \Omega_1$, and so, using also (5.47), we have that for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|(W\varphi_1, \varphi_1)| = \left| \int_{A_r} W(x)\varphi_1^2(x)\,dx + \tilde{\mathcal{O}}(e^{-2(s_1+r)/\hbar}) \right| \tag{5.51}$$

$$\geqslant \frac{1}{C_\epsilon} e^{-(2s_1+\epsilon)/\hbar} \tag{5.52}$$

for small $\hbar$. Moreover, choosing $\epsilon > 0$ such that $s_2 > s_1 + \epsilon$ then

$$(W\varphi_2, \varphi_2) = \tilde{\mathcal{O}}(e^{-2(s_1+\epsilon)/\hbar}) \qquad \text{as } \hbar \to 0. \tag{5.53}$$

Thus, in the sub-critical case **S**, from theorem 3, remark 5, lemma 11 and (5.49) follows that there exists $\epsilon > 0$ such that

$$\begin{aligned} |p^\nu| &\geqslant \frac{|\nu| e^{-(2s_1+\epsilon)/\hbar}}{2C_\epsilon |\mu^\nu(\hbar)| e^{-S_0/\hbar}} \geqslant |\nu| \tilde{C}_\epsilon \hbar^{n/2-1} e^{(S_0 - 2s_1 - \epsilon)/\hbar} \\ &\geqslant |\nu| \tilde{C}_\epsilon \hbar^{n/2-1} e^{a/\hbar} \end{aligned} \tag{5.54}$$

for $\hbar$ small enough and for some positive constant $\tilde{C}_\epsilon$. Therefore, p^ν exponentially goes to infinity as $\hbar$ goes to zero since $a + \hbar \ln|\nu| > 0$ and so we have localization as $\hbar \to 0$.

Finally, in the hypercritical case **H**, (5.54) is replaced by

$$|p^\nu| \geqslant |\nu| \frac{\mathrm{e}^{-(2s_1+\epsilon)/\hbar)}}{2C_\epsilon|\beta^\nu|} = |\nu| \frac{\mathrm{e}^{-(2s_1+\epsilon)/\hbar)}}{2C_\epsilon \mathcal{O}(\mathrm{e}^{-(S_0+\epsilon_0/2)/\hbar)}} \qquad \text{as } \hbar \to 0 \tag{5.55}$$

and then the localization result still follows in the same way since $S_0 = 2S > 2s_1$. □

Remark 13. Let us stress that when we have localization, i.e. $s_1 < \frac{1}{2}S_0$, the splitting between the two resonances is still essentially real as for $\nu = 0$ and moreover it increases when the asymmetric perturbation is switched on:

$$|E_2^\nu(\hbar) - E_1^\nu(\hbar)| = \left|\sqrt{(\alpha_1^\nu - \alpha_2^\nu)^2 + (2\beta^\nu)^2}\right| \geqslant C\mathrm{e}^{-(2s_1+\epsilon)/\hbar} \tag{5.56}$$

as $\hbar$ goes to zero.

Remark 14. Let us stress that the results given in theorem 12 are still true for a perturbation W such that the distances between the wells and the support of W coincide:

$$s_1 = s_2 \tag{5.57}$$

and where the breakdown of the symmetry is performed by assuming

$$\mathcal{S}W = -W\mathcal{S}. \tag{5.58}$$

In such a case, the proof of the theorem 12 is essentially the same where now the behaviour (5.52) holds for $(W\varphi_2, \varphi_2)$ too and so in (5.54) we have C_ϵ instead of $2C_\epsilon$.

5.3. *Symmetrical double volcano with external perturbation*

We consider now the semiclassical Schrödinger operator P_ν where the bounded perturbation which causes the breakdown of the symmetry is *external* (see figure 3). That is

$$\mathcal{W} \cap \bar{\ddot{\mathrm{O}}} = \emptyset. \tag{5.59}$$

In order to discuss a delocalization criterion let us recall that the eigenfunctions $\varphi_\ell(x;\hbar)$ have the behaviour:

$$\varphi_\ell(x;\hbar) = \tilde{\mathcal{O}}(\mathrm{e}^{-S/\hbar}) \qquad \forall x \in U \qquad \text{as } \hbar \to 0 \tag{5.60}$$

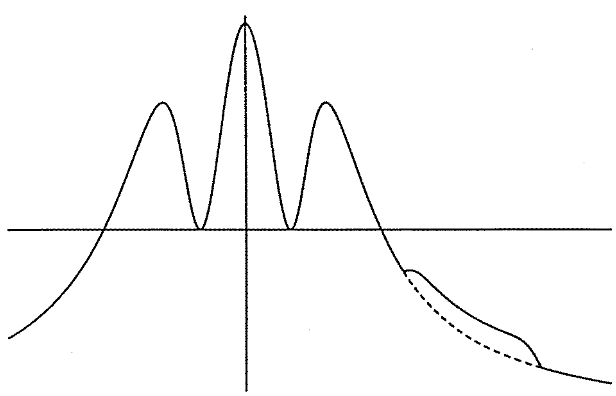

Figure 3. Graph of a one-dimensional double volcano potential with external perturbation (broken curve denotes the symmetric unperturbed double volcano).

and thus

$$z_\ell^\nu(\hbar) = \tilde z(\hbar) + \nu\tilde{\mathcal{O}}(e^{-2S/\hbar}) \qquad \text{as } \hbar \to 0 \tag{5.61}$$

by theorem 3. Therefore, we can state the following delocalization result:

Theorem 15. Let W be a bounded external perturbation. Then, in the sub-critical case **S** the resonant state of P_ν associated to $E_\ell^\nu(\hbar)$, $\ell = 1, 2$, is delocalized for any ν.

Proof. From lemma 11 and (5.61) it immediately follows that

$$p^\nu = \nu\tilde{\mathcal{O}}(e^{-(2S-S_0)/\hbar}) \qquad \text{as } \hbar \to 0 \tag{5.62}$$

which exponentially goes to zero since $2S > S_0$ in the sub-critical case **S**. □

Now, in order to give a criterion of localization we impose some restriction on the class of external perturbations W admitted. More precisely, we assume the perturbation given by $W = w\mathbb{1}_{\mathcal{W}}$ where $w(x)$ is a positive real-valued $C^\infty(\mathbb{R}^n)$ function, $\mathbb{1}_{\mathcal{W}}$ is the characteristic function on $\mathcal{W}$ and $\mathcal{W}$ is a compact subset of $\mathbb{R}^n$ with smooth boundary and satisfying (5.59). Moreover, for the sake of simplicity, let us assume that there exists a unique geodesic γ_ℓ connecting each well $\{x_\ell\}$ with $\partial\ddot{O}$ and let $\tilde x_\ell \in \partial\ddot{O}$ be the endpoint of this geodesic. Then $\tilde x_\ell$ is a point of type 1 in the sense of [8]: that is $\tilde x_\ell \in \overline{B(x_\ell, S)} \cap \partial\ddot{O}$. Let $\tilde\gamma_\ell$ be the bicharacteristic curve starting from $\tilde x_\ell$ and contained in U. We now introduce a system of local coordinates $x = (q', q_n)$ where $q_n \in \mathbb{R}^+$ is the coordinate on $\tilde\gamma_\ell$ and where $q' = (q_1, \ldots, q_{n-1}) \in \mathbb{R}^{n-1}$ is the system of coordinates of the hyperplane orthogonal to $\tilde\gamma_\ell$ at q_n. Then the eigenfunction φ_ℓ has the following form for x belonging to a neighbourhood $\mathcal{V}_\ell$ of $\{\tilde\gamma_\ell(s) : 0 < s \leqslant s^\star\}$ for $s^\star$ small enough (see section 10 in [8]):

$$\varphi_\ell(x;\hbar) = e^{-S/\hbar}\tilde\varphi_\ell(x;\hbar) \qquad \tilde\varphi_\ell(x;\hbar) = a_\ell(x;\hbar)\hbar^{-n/4}e^{-f_\ell(x)/\hbar} \tag{5.63}$$

where $f_\ell = f_\ell^1 + i f_\ell^2$ with f_ℓ^1 and f_ℓ^2 real and analytic and a_ℓ is the realization of an elliptic symbol. Moreover we have that:

$$0 \leqslant f_\ell^1(x) \leqslant C_\ell|q'|^2 \qquad f_\ell^1(x) > 0 \text{ if } q' \neq 0 \tag{5.64}$$

and

$$f_\ell^2(x) = c_\ell(x)[q_n + d_\ell(q')]^{3/2} \qquad c_\ell(x) > C_\ell \qquad |d_\ell(q')| \leqslant C_\ell|q'|^2 \tag{5.65}$$

for some positive constant C_ℓ. From these behaviours and by assuming that the support of W is contained in $\mathcal{V}_1$ and disjoint from $\mathcal{V}_2$ we have that

$$(W\varphi_2, \varphi_2) = \mathcal{O}(e^{-2(S+\epsilon_0)/\hbar}) \qquad \text{as } \hbar \to 0 \tag{5.66}$$

for some $\epsilon_0 > 0$ since (4.13), and

$$(W\varphi_1, \varphi_1) = \hbar^{-n/2}e^{-2S/\hbar}\int_{\mathcal{W}} w(x)a_1^2(x;\hbar)e^{-2f_1(x)/\hbar}\,dx. \tag{5.67}$$

In order to estimate this integral we make the further assumption that (see figure 4)

$$\partial\mathcal{W} \cap \tilde\gamma_1 = \{\alpha_1, \ldots, \alpha_N\} \qquad \text{transversely} \tag{5.68}$$

and we stress that $\nabla f_1 \neq 0$ on $\mathcal{W}$ since (5.65). Thus, by integrating (5.67) by parts m times we obtain for $\hbar$ small enough

$$(W\varphi_1, \varphi_1) = e^{-2S/\hbar}\hbar^{-n/2}\left[\int_{\partial\mathcal{W}} G_m e^{-2f_1/\hbar}\,d\Gamma + \mathcal{O}(\hbar^{m+1})\right] \tag{5.69}$$

where

$$G_m := \sum_{s=0}^{m-1}(-\hbar)^{s+1}g_s \tag{5.70}$$

and $g_s = \boldsymbol{u}_s \cdot \boldsymbol{n}$, $\boldsymbol{n}$ is the unit exterior normal to $\partial\mathcal{W}$, $\boldsymbol{u}_0 = -\frac{1}{2}\frac{\nabla f_1}{|\nabla f_1|^2} w a_1^2$ and $\boldsymbol{u}_s$ is defined as $\boldsymbol{u}_s = -\frac{1}{2}\frac{\nabla f_1}{|\nabla f_1|^2}\nabla\cdot\boldsymbol{u}_{s-1}$. Finally, since $f_1^1|_{\partial\mathcal{W}}$ take its minimum value in correspondence of the points $\{\alpha_1, \ldots, \alpha_N\}$ we obtain the asymptotic behaviour of (5.67) as $\hbar$ goes to zero. More precisely, by assuming for the sake of simplicity that the unit external normal $\boldsymbol{n}$ to $\partial\mathcal{W}$ in α_j is tangent to the bicharacteristic curve $\tilde{\gamma}_\ell$ and that $f_1^1|_{\partial\mathcal{W}}$ has non-degenerate minima in the points α_j, it follows that [section XI.5, 14]:

$$\int_{\partial\mathcal{W}} G_m \mathrm{e}^{-2f_1/\hbar}\,\mathrm{d}\Gamma = \hbar^{(n+1)/2}\sum_{j=1}^{N}\mathrm{e}^{-ic_j/\hbar} b_j(1+\mathcal{O}(\hbar)) \tag{5.71}$$

where c_j are real and positive constants given by

$$c_j = 2f_1^2(\alpha_j) = 2[q_n(\alpha_j)]^{3/2} c(\alpha_j) \tag{5.72}$$

and b_j are complex constants different from zero given by

$$b_j = (2\pi)^{(n-1)/2}|\det A_j|^{-1/2} g_0(\alpha_j)\mathrm{e}^{\mathrm{i}\pi\sigma_j/4} \tag{5.73}$$

where A_j is the Hessian matrix of f_1^1 in α_j and σ_j is the signature of A_j.

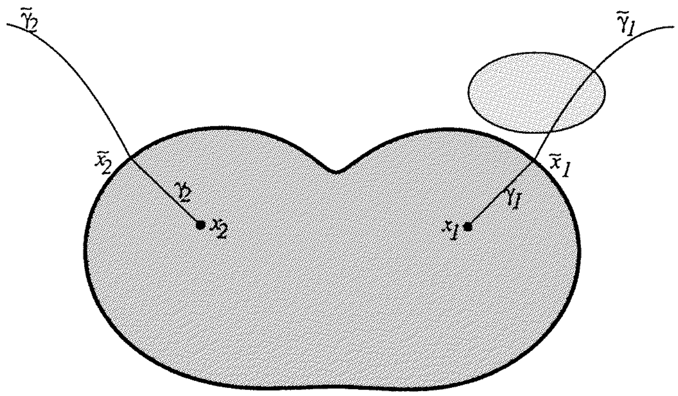

Figure 4. In the external perturbation case the bicharacteristic curve $\tilde{\gamma}_1$ transversally crosses the support $\mathcal{W}$ of the perturbation.

Finally, choosing $m > (n-1)/2$, we obtain:

$$(W\varphi_1, \varphi_1) = \mathrm{e}^{-2S/\hbar} q_1(\hbar) \tag{5.74}$$

where

$$q_1(\hbar) := \hbar^{1/2}\sum_{j=1}^{N}\mathrm{e}^{-\mathrm{i}c_j/\hbar} b_j(1+\mathcal{O}(\hbar)) \qquad \text{as } \hbar \to 0. \tag{5.75}$$

Thus we have proved the following:

Lemma 16. Let $W = w\mathbb{1}_{\mathcal{W}}$ be a bounded external and asymmetric perturbation as defined above, let $\tilde{\gamma}_\ell$ be the bicharacteristic curve and let $\mathcal{V}_\ell$ be a small enough neighbourhood of $\{\tilde{\gamma}_\ell(s) : 0 < s \leqslant s^\star\}$ for $s^\star$ small enough. If $\mathcal{W} \subset \mathcal{V}_1$, $\mathcal{W} \cap \mathcal{V}_2 = \emptyset$ and $\partial\mathcal{W} \cap \tilde{\gamma}_1 = \{\alpha_1, \ldots, \alpha_N\}$ transversely then there exists $\lambda > 0$ such that

$$z_2^\nu(\hbar) = \tilde{z}(\hbar) + \mathcal{O}(\mathrm{e}^{-(2S+\lambda)/\hbar}) \qquad \text{as } \hbar \to 0 \tag{5.76}$$

and

$$z_1^\nu(\hbar) = \tilde{z}(\hbar) + \nu\mathrm{e}^{-2S/\hbar} q_1(\hbar) + \nu\tilde{\mathcal{O}}(\mathrm{e}^{-9S/4\hbar}) \qquad \text{as } \hbar \to 0. \tag{5.77}$$

Proof. The proof follows from the above computations, from theorem 3 and from the bound

$$|((\widetilde{P_t^\nu} - z)^{-1} W\varphi_1, W\varphi_1)| = |e^{-2S/\bar{h}}((\widetilde{P_t^\nu} - z)^{-1} W\tilde{\varphi}_1, W\tilde{\varphi}_1)| \tag{5.78}$$

$$= \mathcal{O}(e^{-2S/\bar{h}} e^{2C_2 t/\bar{h}}). \tag{5.79}$$

Then, since $|\nu| \leqslant e^{-\tilde{C}t^\star/\bar{h}}$ the above behaviour holds. □

Remark 17. From lemma 16 and since the coefficients b_j are different from zero it follows that $q_1(\bar{h})$ admits a strongly oscillating behaviour as $\bar{h}$ goes to zero. Let $m \geqslant 0$ and $0 < c$ be fixed and let $M := M(m, c)$ be the set of $0 < \bar{h} < \bar{h}_0$ such that

$$c\bar{h}^{m+1/2} \leqslant |q_1(\bar{h})|. \tag{5.80}$$

From (5.75) we have that $0 \in \overline{M}$ since the coefficients b_j are different from zero. Moreover, for a large class of models we have that $M = (0, \bar{h}_0]$ for some $\bar{h}_0 > 0$ and some $m \geqslant 0$ and $c > 0$.

Finally, we can state the following theorem:

Theorem 18. Let $W := w\mathbb{1}_{\mathcal{W}}$ be a bounded external and asymmetric perturbation such that $\mathcal{W}$ is a compact subset of $\mathbb{R}^n$ with smooth boundary, $w(x)$ is a positive C^∞ function, $\mathcal{W} \cap \mathcal{V}_2 = \emptyset$, $\mathcal{W} \subset \mathcal{V}_1$ and $\partial\mathcal{W} \cap \tilde{\gamma}_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_N\}$ transversely. Let $\bar{h} \in (0, \bar{h}_0]$ with $\bar{h}_0$ small enough. Then, in the hypercritical case **H** there exists $\epsilon_0 > 0$ such that for any $m \geqslant 0$ and $c > 0$ and for any $\bar{h} \in M(m, c)$ and any $|\nu| \leqslant e^{-\tilde{C}t^\star/\bar{h}}$ such that $-\bar{h}\ln|\nu| < \frac{1}{2}\epsilon_0$ the resonant state of P_ν associated to $E_\ell^\nu(\bar{h})$, $\ell = 1, 2$, is localized.

Proof. Since lemma 11 and lemma 16 we have that there exists $\epsilon_0 > 0$ and $\lambda > 0$ such that

$$p^\nu = \nu \frac{q_1(\bar{h})e^{-2S/\bar{h}} + \mathcal{O}(e^{-(2S+\lambda)/\bar{h}})}{\mathcal{O}(e^{-(S_0+\epsilon_0/2)/\bar{h}})} \qquad \text{as } \bar{h} \to 0 \tag{5.81}$$

where $S_0 = 2S$ in the hypercritical case **H**. Thus p^ν exponentially goes to infinity as $\bar{h}$ goes to zero in $M(m, c)$ since $-\bar{h}\ln|\nu| < \frac{1}{2}\epsilon_0$ and $|q_1(\bar{h})| \geqslant c\bar{h}^{m+1/2}$. □

Remark 19. Let us stress that we can always impose $\tilde{C}t^\star < \frac{1}{2}\epsilon_0$ by means of a possible reduction of $t^\star$ because ϵ_0 does not depend on t. Moreover, we have that the same localization result is still true even in the following case:

$$\mathcal{S}W = -W\mathcal{S} \tag{5.82}$$

and with $\partial\mathcal{W}$ that transversely intersects the bicharacteristic curves and $\mathcal{W} \subset \mathcal{V}_1 \cup \mathcal{V}_2$.

Remark 20. Let us stress that when we have the localization condition the splitting between the resonances is no more essentially real as in the internal perturbation case and it is given by

$$E_2^\nu(\bar{h}) - E_1^\nu(\bar{h}) = (\alpha_2^\nu - \alpha_1^\nu)\sqrt{1 + (p^\nu)^{-2}}$$

$$= -\nu[q_1(\bar{h})e^{-2S/\bar{h}} + \mathcal{O}(e^{-(2S+\lambda)/\bar{h}})](1 + \mathcal{O}((p^\nu)^{-2}))$$

for some $\lambda > 0$, from (5.21), lemmas 11 and 16. Thus, the absolute value of the splitting has the same magnitude of the one of the unperturbed case (since $S_0 = 2S$) but with the following peculiarity: there is also an imaginary part of the splitting which could be dominant with respect to the real part and moreover, for some value of $\bar{h}$, the splitting is purely imaginary.

Remark 21. Actually the fact that d($\mathcal{W}$,Ö) (d denotes the Euclidean distance on $\mathbb{R}^n$) is small is not essential: the same proof works if d($\mathcal{W}$,Ö) is large under the condition that the bicharacteristic curve $\tilde{\gamma}_\ell$ which intersects $\mathcal{W}$ does not develop caustics (inside $\mathcal{W}$ and between $\mathcal{W}$ and Ö). If it does, the result is still probably true, and could be proven by using an FBI transformation which in some sense eliminates the caustics in the same spirit as [10].

5.4. Stark double volcano with external perturbation

Let us consider now the case of a further perturbation given by means of a bounded Stark effect, that is:

$$P_{\nu,f} := P_f + \nu W \qquad P_f := P + fV_S \tag{5.83}$$

where $|f| \leqslant C\overline{h}^2$ for some $C > 0$ and V_S is a real-analytic Stark type potential such that $V_S(x_1) \neq V_S(x_2)$, for instance $V_S(x_1) = -V_S(x_2) \neq 0$, and V_S bounded; for instance

$$V_S(x) = \frac{x^1}{\sqrt{1+(x^1)^2}}. \tag{5.84}$$

Since f is small together with $\overline{h}$ we don't care about the fact that the shape of the island associated to the potential of P_f could be slightly different from the one of the island associated to the potential of P. Similarly, we can identify the Agmon distance of P_f with the one of P. Let us fix ourselves in the hypercritical case **H** and we make the same assumptions on W as in theorem 18. One can check that the single well resonance of $P_{\nu,f,\ell} := P_{\nu,f} + V_\ell$ is given by

$$z_\ell^{\nu,f}(\overline{h}) = z_\ell^{\nu}(\overline{h}) + fV_S(x_\ell) + \mathcal{O}(f^2). \tag{5.85}$$

From this and from lemma 11 the two resonances $E_\ell^{\nu,f}(\overline{h})$ of $P_{\nu,f}$ are given by

$$E_\ell^{\nu,f}(\overline{h}) = \frac{z_1^{\nu,f}(\overline{h}) + z_2^{\nu,f}(\overline{h})}{2} + (-1)^\ell \beta^{\nu,f}\sqrt{1+(p^{\nu,f})^2} + \mathcal{O}(\mathrm{e}^{-(S_0+2\epsilon_0)/\overline{h}})$$

for some $\epsilon_0 > 0$, where

$$\begin{aligned} p^{\nu,f} &= \frac{z_1^{\nu,f}(\overline{h}) - z_2^{\nu,f}(\overline{h})}{2\beta^{\nu,f}} \\ &= \frac{z_1^{\nu}(\overline{h}) - z_2^{\nu}(\overline{h}) + 2fV_S(x_1) + \mathcal{O}(f^2)}{2\beta^{\nu}} \\ &= \frac{\nu\overline{h}^{-n/2}q_1(\overline{h})\mathrm{e}^{-2S/\overline{h}} + 2fV_S(x_1) + \mathcal{O}(f^2)}{\mathcal{O}(\mathrm{e}^{-(S_0+\epsilon_0/2)/\overline{h}})}. \end{aligned}$$

We have the following picture: for *large* f, that is $\ln|f| > -S_0/\overline{h}$, we always have localization for any ν. For *small* f, that is $\ln|f| < -S_0/\overline{h}$, the localization criterion is the one given in theorem 18 as for as $f = 0$, i.e. we have localization for any ν such that $-\overline{h}\ln|\nu| < \frac{1}{2}\epsilon_0$.

Now, let

$$I := I(c,m) = \{\overline{h} > 0 : |\Im q_1(\overline{h})| > c\overline{h}^{m+1/2}\} \qquad c > 0,\ m \geqslant 0. \tag{5.86}$$

From lemma 16 we have that $0 \in \overline{I(c,m)}$ for any c and m and that

$$\lim_{\overline{h}_0 \to 0} \frac{\lambda[(0,\overline{h}_0) \cap I(c,m)]}{\overline{h}_0} = 1 \tag{5.87}$$

for any $c > 0$ and $m \geqslant 2$, where λ denotes the usual Lebesgue measure on $\mathbb{R}$. For fixed c and m let us now consider $\overline{h} \in I(c, m)$ fixed and small enough. Then, there exists a real value of $\tilde{f} := \tilde{f}(\nu, \overline{h})$, with $\ln|\tilde{f}| \leqslant -S_0/\overline{h}$, such that

$$\Re(z_1^{\nu,\tilde{f}} - z_2^{\nu,\tilde{f}}) = 0 \tag{5.88}$$

and so $z_1^{\nu,\tilde{f}} - z_2^{\nu,\tilde{f}}$ is purely imaginary with imaginary part given by

$$\Im(z_1^{\nu,\tilde{f}} - z_2^{\nu,\tilde{f}}) = \nu[\mathrm{e}^{-2S/\overline{h}}\Im q_1(\overline{h}) + \mathcal{O}(\mathrm{e}^{-(2S+\lambda)/\overline{h}})] \tag{5.89}$$

for some $\lambda > 0$, since lemma 16. Thus,

$$|p^{\nu,\tilde{f}}| \geqslant \nu \frac{C\overline{h}^{m+1/2-n/2}\mathrm{e}^{-2S/\overline{h}}}{\mathcal{O}(\mathrm{e}^{-(S_0+\epsilon_0/2)/\overline{h}})} \tag{5.90}$$

for some C, which is much greater than 1 since $\epsilon_0 > 0$, $-\overline{h}\ln|\nu| < \frac{1}{2}\epsilon_0$ and $\overline{h} \ll 1$. Then, for $\overline{h} \in I(c, m)$ and f near to $\tilde{f}$ we still have localization as in the case of absence of the Stark effect but with the following peculiarity: the splitting between the resonances is purely imaginary, i.e.

$$E_1^{\nu,\tilde{f}}(\overline{h}) - E_2^{\nu,\tilde{f}}(\overline{h}) = -\mathrm{i}\nu[\Im q_1 \mathrm{e}^{-2S/\overline{h}} + \mathcal{O}(\mathrm{e}^{-(2S+\lambda)/\overline{h}})](1 + \mathcal{O}((p^{\nu,\tilde{f}})^{-2})).$$

We can summarize this result as follows:

Theorem 22. Let the hypotheses of theorem 18 be satisfied. Then, in the hypercritical case **H** for any $\overline{h} \in I(c, m)$ and ν small enough there exists $f := f(\nu, \overline{h})$ such that we have localization for $P_{\nu,f}$ and the real part of the splitting between the two resonances $E_1^{\nu,f}(\overline{h})$ and $E_2^{\nu,f}(\overline{h})$ is zero.

In such a case we have the splitting instability when the symmetry of the double-well potential is broken by means of the external perturbation W and the Stark effect.

Appendix

In this appendix we introduce the Hilbert spaces $\mathcal{H}_s^t$ of Helffer–Sjöstrand [8] and we state some basic results about them. We assume that V is a real-valued potential satisfying to the following hypotheses:

Hypothesis 1. $V \in C^\infty(\mathbb{R}^n)$ and it admits an analytic extension outside a real compact set of $\mathbb{R}^n$; more precisely V is holomorphic in the region

$$\mathcal{R} := \{z \in \mathbb{C}^n : |\Re z| > R \qquad \text{and} \qquad |\Im z| < \delta|\Re z|\} \tag{A.1}$$

for some $\delta > 0$ and $R > 0$.

Hypothesis 2. There exists an integer $k \geqslant 0$ and a positive constant C such that

$$|V(z)| \leqslant C|\Re z|^k \qquad \forall z \in \mathcal{R}. \tag{A.2}$$

Hypothesis 3. There exists a real-valued function $G \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)$ such that

$$\partial_x^\alpha \partial_\xi^\beta G(x, \xi) = \mathcal{O}(|x|^{1-|\alpha|} r(x, \xi)^{1-|\beta|}) \qquad \forall \alpha, \beta \in \mathbb{N} \tag{A.3}$$

where $r(x, \xi) := [\langle \Re x\rangle^k + \langle \Re\xi\rangle^2]^{1/2}$, $\langle u\rangle := [1 + |u|^2]^{1/2}$, and such that

$$2\xi \cdot \nabla_x G - \nabla_x V \cdot \nabla_\xi G \geqslant C|x|^k \tag{A.4}$$

for any (x, ξ) belonging to $\{(x, \xi) \in U \times R_\xi^n : \xi^2 + V(x) = 0\}$, for some positive constant C.

One can see that the function $-G(x,-\xi)$ also satisfies hypothesis 3, and therefore the same is true for $G(x,\xi)-G(x,-\xi)$. Thus, without loss of generality, we can assume from now on that $G(x,\xi)=-G(x,-\xi)$, and also that for any compact K the set

$$\cup_{x\in K}\operatorname{supp} G(x,\cdot) \tag{A.5}$$

is compact.

We define, for $t\in\mathbb{R}$:

$$\Lambda_{tG} := \{(x,\xi)\in\mathbb{C}^{2n} : \Im x = t\nabla_\xi G(\Re x,\Re\xi) \qquad \Im\xi = -t\nabla_x G(\Re x,\Re\xi)\}$$

and for $u\in C_0^\infty(\mathbb{R}^n)$

$$Tu(x,\xi;\hbar) := \int_{\mathbb{R}^n} \mathrm{e}^{[\mathrm{i}(x-y)\xi - \frac{r(x,\xi)}{\langle\Re x\rangle}(x-y)^2]/\hbar}\boldsymbol{a}\chi\left(\frac{\Re x - y}{\langle\Re x\rangle}\right)u(y)\,\mathrm{d}y$$

where $\chi\in C_0^\infty(\mathbb{R}^n)$ has value 1 near 0 and is supported in a sufficiently small neighbourhood of 0 (say the ball with centre 0 and of diameter $\delta\leqslant 1/2$) and

$$\boldsymbol{a} := \boldsymbol{a}(x,y,\xi) = \langle\xi\rangle^{n/4}\langle x\rangle^{-n/4}\left(1,\frac{y}{\langle x\rangle}\right). \tag{A.6}$$

Then $\mathcal{H}_s^t := \mathcal{H}_s^t(G)$ is defined as the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm:

$$\|u\|_{\mathcal{H}_s^t(G)} := \|r(x,\xi)^s Tu(x,\xi;\hbar)\|_{L^2(\Lambda_{tG};\mathrm{e}^{-2tH/\hbar}\,\mathrm{d}\Re x\,\mathrm{d}\Re\xi)} \tag{A.7}$$

where H is the real-valued function defined as

$$H(x,\xi) := G(\Re x,\Re\xi) - \Re\xi\cdot\nabla_\xi G(\Re x,\Re\xi). \tag{A.8}$$

It allows from the definition that $\mathcal{H}_s^t\subset\mathcal{H}_{s'}^t$ if $s>s'$, and that $u\in\mathcal{H}_0^t$ if, and only if, $\bar{u}\in\mathcal{H}_0^{-t}$. Moreover, the $L^2(\mathbb{R}^n)$ scalar product on $C_0^\infty(\mathbb{R}^n)\times C_0^\infty(\mathbb{R}^n)$ can be extended to a continuous map from $\mathcal{H}_0^t\times\mathcal{H}_0^{-t}$ to $\mathbb{C}$. In particular, the quantity $\langle u,\bar{u}\rangle_{\mathcal{H}_0^t\times\mathcal{H}_0^{-t}}$ is well defined for $u\in\mathcal{H}_0^t$.

Now, denoting $\mu(x,\xi)=\frac{r(x,\xi)}{\langle\Re x\rangle}$, we have for $(x,\xi)\in\Lambda_{tG}$:

$$\begin{aligned}-tH(x,\xi)+\mathrm{i}(x-y)\xi-\mu(x,\xi)(x-y)^2 &= -\tilde{G}_t(x,\xi)-\mathrm{i}\Im x\Im\xi\\ &+\mathrm{i}(\Re x-y)(\Re\xi-2\mu(x,\xi)\Im x)-\mu(x,\xi)\left(\Re x+\frac{\Im\xi}{2\mu(x,\xi)}-y\right)^2\end{aligned}$$

with

$$\tilde{G}_t(x,\xi) := tG(\Re x,\Re\xi)+t^2\mu(x,\xi)(\nabla_\xi G)^2+\frac{t^2}{4\mu(x,\xi)}(\nabla_x G)^2. \tag{A.9}$$

As a consequence, using the change of variables:

$$\begin{cases}\tilde{\xi} = \Re\xi - 2\mu(x,\xi)\Im x\\ \tilde{x} = \Re x + \dfrac{\Im\xi}{2\mu(x,\xi)} = \Re x + \mathcal{O}(t\langle\Re x\rangle)\end{cases} \tag{A.10}$$

we see that the norm $\|\cdot\|_{\mathcal{H}_s^t(G)}$ is uniformly equivalent to the norm:

$$\|u\|_{\mathcal{H}_s^t} := \|r(\tilde{x},\tilde{\xi})^s\tilde{T}u(\tilde{x},\tilde{\xi};\hbar)\|_{L^2(\mathbb{R}^{2n};\mathrm{e}^{-2\tilde{H}_t(\tilde{x},\tilde{\xi})/\hbar}\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{\xi})} \tag{A.11}$$

where $\tilde{T}$ differs from T only by the cut-off χ, and $\tilde{H}_t$ is deduced from $\tilde{G}_t$ by a change of variable (A.10). In particular, by (A.9) we have

$$\tilde{H}_t(\tilde{x},\tilde{\xi}) = tG(\tilde{x},\tilde{\xi})+\mathcal{O}(t^2 r(\tilde{x},\tilde{\xi})\langle\tilde{x}\rangle). \tag{A.12}$$

Now, let $W\in L^\infty(\mathbb{R}^n)$ be compactly supported, and consider the operator (still denoted by W) of multiplication by W acting on $\mathcal{H}_0^t$. We have:

Lemma A1. Let V be a double-well potential as in section 3 satisfying to hypotheses 1–3 and let $\mathcal{H}_s^t$ be the above Hilbert space. Then, for $t > 0$ small enough and $\delta := \operatorname{diam}\operatorname{supp}\chi$ small enough the operator

$$W : \mathcal{H}_0^t \longrightarrow \mathcal{H}_0^t \tag{A.13}$$

is bounded with bound

$$\|W\|_{\mathcal{L}(\mathcal{H}_0^t)} = \mathcal{O}(1) \tag{A.14}$$

uniformly with respect to $\hbar$ if $\operatorname{supp} W \times \mathbb{R}^n \cap \operatorname{supp} G = \emptyset$; or

$$\|W\|_{\mathcal{L}(\mathcal{H}_0^t)} \leqslant C_1 \mathrm{e}^{C_2 t/\hbar} \tag{A.15}$$

for some positive constants C_1 and C_2 independent of t and $\hbar$ if $\operatorname{supp} W \times \mathbb{R}^n \cap \operatorname{supp} G \neq \emptyset$.

Remark A2. Actually, one can prove that (A.14) is true under the more general condition: $WF_a(W) \cap \operatorname{supp} G = \emptyset$, where $WF_a(W)$ denotes the analytic wave-front set of W (see e.g. [13]).

Proof. We work with the norm defined in (A.11). By construction we see that $(\tilde{T} W u)(\tilde{x}, \tilde{\xi})$ vanishes for $\tilde{x}$ outside some compact set K since W has compact support. Moreover, for $\tilde{x} \in K$ we have that $tG(\tilde{x}, \tilde{\xi}) = \tilde{H}_t(\tilde{x}, \tilde{\xi}) = 0$ for $\tilde{\xi}$ outside some other compact set K' since (A.5). Also, taking t and δ small enough, we see that K can be taken inside an arbitrary neighbourhood of $\operatorname{supp} W$. Then, we write

$$\|Wu\|^2_{\mathcal{H}_0^t} = \|\tilde{T} W u\|^2_{L^2(K \times K', \mathrm{e}^{-2\tilde{H}_t/\hbar}\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{\xi})} + \|\tilde{T} W u\|^2_{L^2(K \times K'^C, \mathrm{d}\tilde{x}\,\mathrm{d}\tilde{\xi})} \tag{A.16}$$

where $K'^C := \mathbb{R}^n_{\xi} - K'$ and where $\tilde{H}_t$ is $\mathcal{O}(t)$ on $K \times K'$ since (A.12). Thus, we deduce from (A.16):

$$\|Wu\|_{\mathcal{H}_0^t} \leqslant C_1' \mathrm{e}^{C_2' t/\hbar} \|\tilde{T} W u\|_{L^2(K \times \mathbb{R}^n, \mathrm{d}\tilde{x}\,\mathrm{d}\tilde{\xi})} \tag{A.17}$$

and it is not difficult to show the existence of $\alpha(\hbar)$, $\beta(\hbar) > 0$ such that (see [8])

$$\alpha(\hbar)\|u\|_{L^2(\mathbb{R}^n)} \leqslant \|\tilde{T} u\|_{L^2(\mathbb{R}^{2n})} \leqslant \beta(\hbar)\|u\|_{L^2(\mathbb{R}^n)} \qquad \frac{1}{C} \leqslant \frac{\alpha(\hbar)}{\beta(\hbar)} \leqslant 1 \tag{A.18}$$

for some positive constant C independent of $\hbar$. As a consequence, we deduce from (A.17):

$$\|Wu\|_{\mathcal{H}_0^t} \leqslant \beta(\hbar) C_1' \mathrm{e}^{C_2' t/\hbar} \|W\|_{L^\infty} \|u\|_{L^2(K)} \tag{A.19}$$

and the same argument as for (A.18) gives the existence of a compact set $\tilde{K}$ such that

$$\|u\|_{L^2(K)} = \mathcal{O}\left(\frac{1}{\alpha(\hbar)} \|\tilde{T} u\|_{L^2(\tilde{K} \times \mathbb{R}^n)}\right) \tag{A.20}$$

uniformly. Since also $\tilde{H}_t = \mathcal{O}(t)$ on $\tilde{K} \times \mathbb{R}^n$, we deduce from (A.18), (A.19) and (A.20) that

$$\|Wu\|_{\mathcal{H}_0^t} \leqslant C_1 \mathrm{e}^{C_2 t/\hbar} \|\tilde{T} u\|_{L^2(\tilde{K} \times \mathbb{R}^n, \mathrm{e}^{-2\tilde{H}_t/\hbar}\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{\xi})} \tag{A.21}$$

from which (A.15) follows. Now, if we assume $\operatorname{supp} W \times \mathbb{R}^n \cap \operatorname{supp} G = \emptyset$, we get by taking t and δ sufficiently small: $K \times \mathbb{R}^n \cap \operatorname{supp} G = \emptyset$. Then (A.16) gives

$$\|Wu\|_{\mathcal{H}_0^t} \leqslant \|\tilde{T} W u\|_{L^2(K \times \mathbb{R}^n)} \tag{A.22}$$

and thus as for (A.19):

$$\|Wu\|_{\mathcal{H}_0^t} \leqslant \beta(\hbar) C_1 \|u\|_{L^2(K)}. \tag{A.23}$$

Applying (A.20) and noticing that $\tilde{K}$ can be taken inside an arbitrary neighbourhood of K (so that in particular one can impose $\tilde{K} \times \mathbb{R}^n \cap \operatorname{supp} G = \emptyset$), we get (A.14) as before. □

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